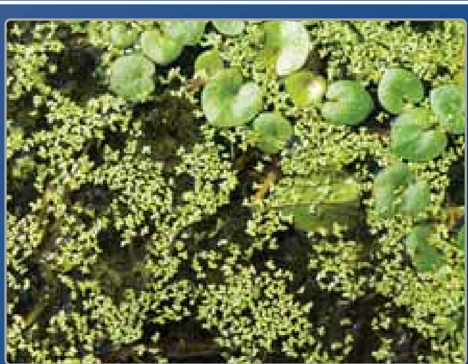
6 Exponential and Logarithmic Functions

Astronaut Health *(p. 347)*

Cooking *(p. 335)*

Recording Studio *(p. 330)*

Tornado Wind Speed *(p. 315)*



Duckweed Growth *(p. 301)*

Maintaining Mathematical Proficiency

Using Exponents

Example 1 Evaluate $\left(-\frac{1}{3}\right)^4$.

$$\left(-\frac{1}{3}\right)^4 = \left(-\frac{1}{3}\right) \cdot \left(-\frac{1}{3}\right) \cdot \left(-\frac{1}{3}\right) \cdot \left(-\frac{1}{3}\right)$$ Rewrite $\left(-\frac{1}{3}\right)^4$ as repeated multiplication.

$$= \left(\frac{1}{9}\right) \cdot \left(-\frac{1}{3}\right) \cdot \left(-\frac{1}{3}\right)$$ Multiply.

$$= \left(-\frac{1}{27}\right) \cdot \left(-\frac{1}{3}\right)$$ Multiply.

$$= \frac{1}{81}$$ Multiply.

Evaluate the expression.

1. $3 \cdot 2^4$

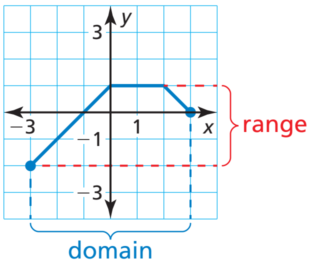
2. $(-2)^5$

3. $-\left(\frac{5}{6}\right)^2$

4. $\left(\frac{3}{4}\right)^3$

Finding the Domain and Range of a Function

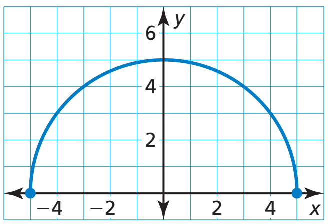
Example 2 Find the domain and range of the function represented by the graph.



▶ The domain is $-3 \leq x \leq 3$.
The range is $-2 \leq y \leq 1$.

Find the domain and range of the function represented by the graph.

5.



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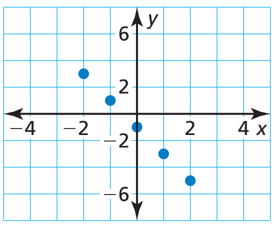

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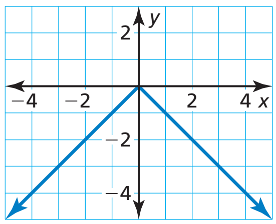

8. ABSTRACT REASONING Consider the expressions -4^n and $(-4)^n$, where n is an integer. For what values of n is each expression negative? positive? Explain your reasoning.

Mathematical Practices

Mathematically proficient students know when it is appropriate to use general methods and shortcuts.

Exponential Models

Core Concept

Consecutive Ratio Test for Exponential Models

Consider a table of values of the given form.

x	0	1	2	3	4	5	6	7	8	9
y	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9

If the consecutive ratios of the y-values are all equal to a common value r, then y can be modeled by an exponential function. When $r > 1$, the model represents exponential *growth*.

$r = \frac{a_{n+1}}{a_n}$ Common ratio

$y = a_0 r^x$ Exponential model

EXAMPLE 1 Modeling Real-Life Data

The table shows the amount A (in dollars) in a savings account over time. Write a model for the amount in the account as a function of time t (in years). Then use the model to find the amount after 10 years.

Year, *t*	0	1	2	3	4	5
Amount, *A*	\$1000	\$1040	\$1081.60	\$1124.86	\$1169.86	\$1216.65

SOLUTION

Begin by determining whether the ratios of consecutive amounts are equal.

$$\frac{1040}{1000} = 1.04, \quad \frac{1081.60}{1040} = 1.04, \quad \frac{1124.86}{1081.60} \approx 1.04, \quad \frac{1169.86}{1124.86} \approx 1.04, \quad \frac{1216.65}{1169.86} \approx 1.04$$

The ratios of consecutive amounts are equal, so the amount A after t years can be modeled by

$$A = 1000(1.04)^t.$$

Using this model, the amount when $t = 10$ is $A = 1000(1.04)^{10} = \$1480.24$.

Monitoring Progress

Determine whether the data can be modeled by an exponential or linear function. Explain your reasoning. Then write the appropriate model and find y when $x = 10$.

1.

x	0	1	2	3	4
y	1	2	4	8	16

2.

x	0	1	2	3	4
y	0	4	8	12	16

3.

x	0	1	2	3	4
y	1	4	7	10	13

4.

x	0	1	2	3	4
y	1	3	9	27	81

Mathematical Practices

Mathematically proficient students ***notice whether calculations are repeated, and look both for general methods and for shortcuts.*** **(MP8)**

Exponential Models

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6.1 Exponential Growth and Decay Functions

Essential Question

Essential Question What are some of the characteristics of the graph of an exponential function?

You can use a graphing calculator to evaluate an exponential function. For example, consider the exponential function $f(x) = 2^x$.

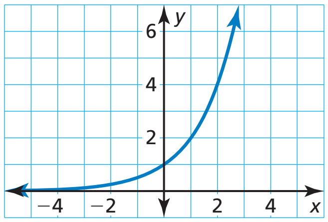
Function Value	Graphing Calculator Keystrokes	Display
$f(-3.1) = 2^{-3.1}$	2 [^] [(–)] 3.1 [ENTER]	0.1166291
$f\left(\frac{2}{3}\right) = 2^{2/3}$	2 [^] [(] 2 [÷] 3 [)] [ENTER]	1.5874011

EXPLORATION 1 Identifying Graphs of Exponential Functions

Work with a partner. Match each exponential function with its graph. Use a table of values to sketch the graph of the function, if necessary.

a. $f(x) = 2^x$ **b.** $f(x) = 3^x$ **c.** $f(x) = 4^x$

d. $f(x) = \left(\frac{1}{2}\right)^x$ **e.** $f(x) = \left(\frac{1}{3}\right)^x$ **f.** $f(x) = \left(\frac{1}{4}\right)^x$

A.


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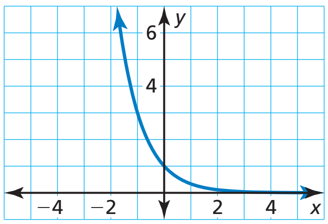

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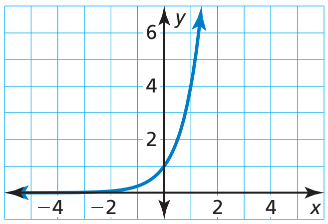

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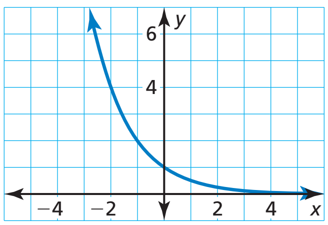

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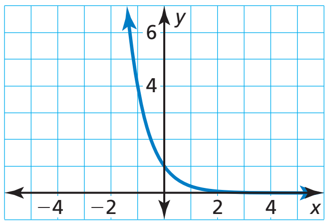

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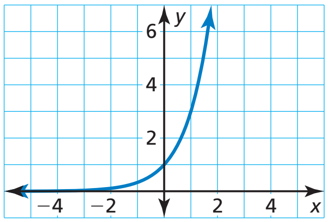


CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to justify your conclusions and communicate them to others.

EXPLORATION 2 Characteristics of Graphs of Exponential Functions

Work with a partner. Use the graphs in Exploration 1 to determine the domain, range, and y-intercept of the graph of $f(x) = b^x$, where b is a positive real number other than 1. Explain your reasoning.

Communicate Your Answer

3. What are some of the characteristics of the graph of an exponential function?

4. In Exploration 2, is it possible for the graph of $f(x) = b^x$ to have an x-intercept? Explain your reasoning.

6.1 Lesson

Core Vocabulary

exponential function, *p. 296*
exponential growth function, *p. 296*
growth factor, *p. 296*
asymptote, *p. 296*
exponential decay function, *p. 296*
decay factor, *p. 296*

Previous
properties of exponents

What You Will Learn

- Graph exponential growth and decay functions.
- Use exponential models to solve real-life problems.

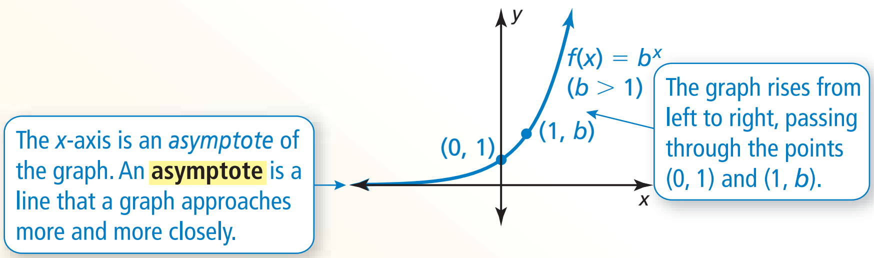
Exponential Growth and Decay Functions

An **exponential function** has the form $y = ab^x$, where $a \neq 0$ and the base b is a positive real number other than 1. If $a > 0$ and $b > 1$, then $y = ab^x$ is an **exponential growth function**, and b is called the **growth factor**. The simplest type of exponential growth function has the form $y = b^x$.

Core Concept

Parent Function for Exponential Growth Functions

The function $f(x) = b^x$, where $b > 1$, is the parent function for the family of exponential growth functions with base b. The graph shows the general shape of an exponential growth function.



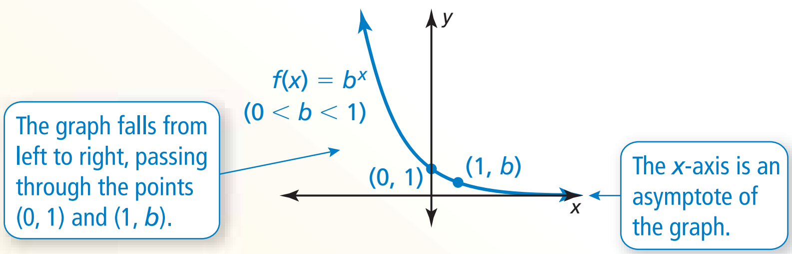
The domain of $f(x) = b^x$ is all real numbers. The range is $y > 0$.

If $a > 0$ and $0 < b < 1$, then $y = ab^x$ is an **exponential decay function**, and b is called the **decay factor**.

Core Concept

Parent Function for Exponential Decay Functions

The function $f(x) = b^x$, where $0 < b < 1$, is the parent function for the family of exponential decay functions with base b. The graph shows the general shape of an exponential decay function.



The domain of $f(x) = b^x$ is all real numbers. The range is $y > 0$.

EXAMPLE 1 Graphing Exponential Growth and Decay Functions

Tell whether each function represents *exponential growth* or *exponential decay*. Then graph the function.

a. $y = 2^x$

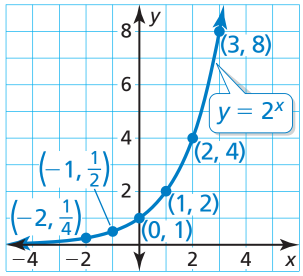
b. $y = \left(\frac{1}{2}\right)^x$

SOLUTION

a. Step 1 Identify the value of the base. The base, 2, is greater than 1, so the function represents exponential growth.

Step 2 Make a table of values.

x	-2	-1	0	1	2	3
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



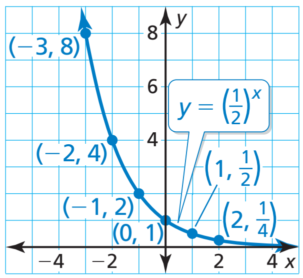
Step 3 Plot the points from the table.

Step 4 Draw, from *left to right*, a smooth curve that begins just above the x-axis, passes through the plotted points, and moves up to the right.

b. Step 1 Identify the value of the base. The base, $\frac{1}{2}$, is greater than 0 and less than 1, so the function represents exponential decay.

Step 2 Make a table of values.

x	-3	-2	-1	0	1	2
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$



Step 3 Plot the points from the table.

Step 4 Draw, from *right to left*, a smooth curve that begins just above the x-axis, passes through the plotted points, and moves up to the left.

Monitoring Progress

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Tell whether the function represents *exponential growth* or *exponential decay*. Then graph the function.

1. $y = 4^x$

2. $y = \left(\frac{2}{3}\right)^x$

3. $f(x) = (0.25)^x$

4. $f(x) = (1.5)^x$

Exponential Models

Some real-life quantities increase or decrease by a fixed percent each year (or some other time period). The amount y of such a quantity after t years can be modeled by one of these equations.

Exponential Growth Model

$$y = a(1 + r)^t$$

Exponential Decay Model

$$y = a(1 - r)^t$$

Note that a is the initial amount and r is the percent increase or decrease written as a decimal. The quantity $1 + r$ is the growth factor, and $1 - r$ is the decay factor.

EXAMPLE 2 Solving a Real-Life Problem

The value of a car y (in thousands of dollars) can be approximated by the model $y = 25(0.85)^t$, where t is the number of years since the car was new.

a. Tell whether the model represents exponential growth or exponential decay.

b. Identify the annual percent increase or decrease in the value of the car.

c. Estimate when the value of the car will be $8000.

REASONING QUANTITATIVELY

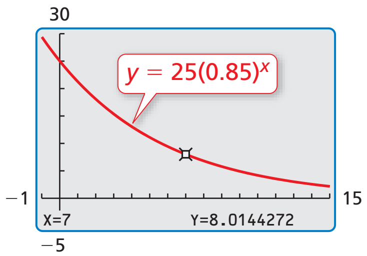
The percent decrease, 15%, tells you how much value the car *loses* each year. The decay factor, 0.85, tells you what fraction of the car's value *remains* each year.

SOLUTION

a. The base, 0.85, is greater than 0 and less than 1, so the model represents exponential decay.

b. Because t is given in years and the decay factor $0.85 = 1 - 0.15$, the annual percent decrease is 0.15, or 15%.

c. Use the *trace* feature of a graphing calculator to determine that $y \approx 8$ when $t = 7$. After 7 years, the value of the car will be about $8000.



EXAMPLE 3 Writing an Exponential Model

In 2000, the world population was about 6.09 billion. During the next 13 years, the world population increased by about 1.18% each year.

a. Write an exponential growth model giving the population y (in billions) t years after 2000. Estimate the world population in 2005.

b. Estimate the year when the world population was 7 billion.

SOLUTION

a. The initial amount is $a = 6.09$, and the percent increase is $r = 0.0118$. So, the exponential growth model is

$y = a(1 + r)^t$ — Write exponential growth model.

$= 6.09(1 + 0.0118)^t$ — Substitute 6.09 for *a* and 0.0118 for *r*.

$= 6.09(1.0118)^t.$ — Simplify.

Using this model, you can estimate the world population in 2005 ($t = 5$) to be $y = 6.09(1.0118)^5 \approx 6.46$ billion.

X	Y1
6	6.5341
7	6.6112
8	6.6892
9	6.7681
10	6.848
11	6.9288
12	7.0106

X=12

b. Use the *table* feature of a graphing calculator to determine that $y \approx 7$ when $t = 12$. So, the world population was about 7 billion in 2012.

Monitoring Progress

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5. **WHAT IF?** In Example 2, the value of the car can be approximated by the model $y = 25(0.9)^t$. Identify the annual percent decrease in the value of the car. Estimate when the value of the car will be $8000.

6. **WHAT IF?** In Example 3, assume the world population increased by 1.5% each year. Write an equation to model this situation. Estimate the year when the world population was 7 billion.

EXAMPLE 4 Rewriting an Exponential Function

The amount y (in grams) of the radioactive isotope chromium-51 remaining after t days is $y = a(0.5)^{t/28}$, where a is the initial amount (in grams). What percent of the chromium-51 decays each day?

SOLUTION

$y = a(0.5)^{t/28}$	Write original function.
$= a[(0.5)^{1/28}]^t$	Power of a Power Property
$\approx a(0.9755)^t$	Evaluate power.
$= a(1 - 0.0245)^t$	Rewrite in form $y = a(1 - r)^t$.

▶ The daily decay rate is about 0.0245, or 2.45%.

Compound interest is interest paid on an initial investment, called the *principal*, and on previously earned interest. Interest earned is often expressed as an *annual* percent, but the interest is usually compounded more than once per year. So, the exponential growth model $y = a(1 + r)^t$ must be modified for compound interest problems.

Core Concept

Compound Interest

Consider an initial principal P deposited in an account that pays interest at an annual rate r (expressed as a decimal), compounded n times per year. The amount A in the account after t years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

EXAMPLE 5 Finding the Balance in an Account

You deposit \$9000 in an account that pays 1.46% annual interest. Find the balance after 3 years when the interest is compounded quarterly.

SOLUTION

With interest compounded quarterly (4 times per year), the balance after 3 years is

$A = P\left(1 + \frac{r}{n}\right)^{nt}$	Write compound interest formula.
$= 9000\left(1 + \frac{0.0146}{4}\right)^{4 \cdot 3}$	$P = 9000$, $r = 0.0146$, $n = 4$, $t = 3$
$\approx 9402.21.$	Use a calculator.

▶ The balance at the end of 3 years is \$9402.21.

Monitoring Progress

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7. The amount y (in grams) of the radioactive isotope iodine-123 remaining after t hours is $y = a(0.5)^{t/13}$, where a is the initial amount (in grams). What percent of the iodine-123 decays each hour?

8. **WHAT IF?** In Example 5, find the balance after 3 years when the interest is compounded daily.

6.1 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. VOCABULARY In the exponential growth model $y = 2.4(1.5)^x$, identify the initial amount, the growth factor, and the percent increase.

2. WHICH ONE DOESN'T BELONG? Which characteristic of an exponential decay function does *not* belong with the other three? Explain your reasoning.

base of 0.8	decay factor of 0.8
decay rate of 20%	80% decrease

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, evaluate the expression for (a) $x = -2$ and (b) $x = 3$.

3. 2^x

4. 4^x

5. $8 \cdot 3^x$

6. $6 \cdot 2^x$

7. $5 + 3^x$

8. $2^x - 2$

In Exercises 9–18, tell whether the function represents *exponential growth* or *exponential decay*. Then graph the function. *(See Example 1.)*

9. $y = 6^x$

10. $y = 7^x$

11. $y = \left(\frac{1}{6}\right)^x$

12. $y = \left(\frac{1}{8}\right)^x$

13. $y = \left(\frac{4}{3}\right)^x$

14. $y = \left(\frac{2}{5}\right)^x$

15. $y = (1.2)^x$

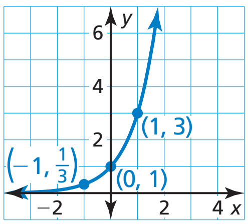
16. $y = (0.75)^x$

17. $y = (0.6)^x$

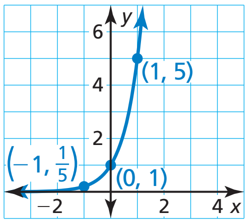
18. $y = (1.8)^x$

ANALYZING RELATIONSHIPS **In Exercises 19 and 20, use the graph of $f(x) = b^x$ to identify the value of the base b.**

19.



20.



21. MODELING WITH MATHEMATICS The value of a mountain bike y (in dollars) can be approximated by the model $y = 200(0.75)^t$, where t is the number of years since the bike was new. *(See Example 2.)*

a. Tell whether the model represents exponential growth or exponential decay.

b. Identify the annual percent increase or decrease in the value of the bike.

c. Estimate when the value of the bike will be \$50.

22. MODELING WITH MATHEMATICS The population P (in thousands) of Austin, Texas, during a recent decade can be approximated by $y = 494.29(1.03)^t$, where t is the number of years since the beginning of the decade.

a. Tell whether the model represents exponential growth or exponential decay.

b. Identify the annual percent increase or decrease in population.

c. Estimate when the population was about 590,000.

23. MODELING WITH MATHEMATICS In 2006, there were approximately 233 million cell phone subscribers in the United States. During the next 4 years, the number of cell phone subscribers increased by about 6% each year. *(See Example 3.)*

a. Write an exponential growth model giving the number of cell phone subscribers y (in millions) t years after 2006. Estimate the number of cell phone subscribers in 2008.

b. Estimate the year when the number of cell phone subscribers was about 278 million.

24. MODELING WITH MATHEMATICS You take a 325 milligram dosage of ibuprofen. During each subsequent hour, the amount of medication in your bloodstream decreases by about 29% each hour.

a. Write an exponential decay model giving the amount y (in milligrams) of ibuprofen in your bloodstream t hours after the initial dose.

b. Estimate how long it takes for you to have 100 milligrams of ibuprofen in your bloodstream.

JUSTIFYING STEPS In Exercises 25 and 26, justify each step in rewriting the exponential function.

25.

$y = a(3)^{t/14}$	Write original function.
$= a[(3)^{1/14}]^t$	
$\approx a(1.0816)^t$	
$= a(1 + 0.0816)^t$	

26.

$y = a(0.1)^{t/3}$	Write original function.
$= a[(0.1)^{1/3}]^t$	
$\approx a(0.4642)^t$	
$= a(1 - 0.5358)^t$	

27. PROBLEM SOLVING When a plant or animal dies, it stops acquiring carbon-14 from the atmosphere. The amount y (in grams) of carbon-14 in the body of an organism after t years is $y = a(0.5)^{t/5730}$, where a is the initial amount (in grams). What percent of the carbon-14 is released each year? *(See Example 4.)*

28. PROBLEM SOLVING The number y of duckweed fronds in a pond after t days is $y = a(1230.25)^{t/16}$, where a is the initial number of fronds. By what percent does the duckweed increase each day?

In Exercises 29–36, rewrite the function in the form $y = a(1 + r)^t$ or $y = a(1 - r)^t$. Then state the growth or decay rate.

29. $y = a(2)^{t/3}$

30. $y = a(4)^{t/6}$

31. $y = a(0.5)^{t/12}$

32. $y = a(0.25)^{t/9}$

33. $y = a\left(\frac{2}{3}\right)^{t/10}$

34. $y = a\left(\frac{5}{4}\right)^{t/22}$

35. $y = a(2)^{8t}$

36. $y = a\left(\frac{1}{3}\right)^{3t}$

37. PROBLEM SOLVING You deposit \$5000 in an account that pays 2.25% annual interest. Find the balance after 5 years when the interest is compounded quarterly. *(See Example 5.)*

38. DRAWING CONCLUSIONS You deposit \$2200 into three separate bank accounts that each pay 3% annual interest. How much interest does each account earn after 6 years?

Account	Compounding	Balance after 6 years
1	quarterly	
2	monthly	
3	daily	

39. ERROR ANALYSIS You invest \$500 in the stock of a company. The value of the stock decreases 2% each year. Describe and correct the error in writing a model for the value of the stock after t years.

✗

$$y = \left(\text{Initial amount}\right)\left(\text{Decay factor}\right)^t$$

$$y = 500(0.02)^t$$

40. ERROR ANALYSIS You deposit \$250 in an account that pays 1.25% annual interest. Describe and correct the error in finding the balance after 3 years when the interest is compounded quarterly.

✗

$$A = 250\left(1 + \frac{1.25}{4}\right)^{4 \cdot 3}$$

$$A = \$6533.29$$

In Exercises 41–44, use the given information to find the amount A in the account earning compound interest after 6 years when the principal is \$3500.

41. $r = 2.16\%$, compounded quarterly

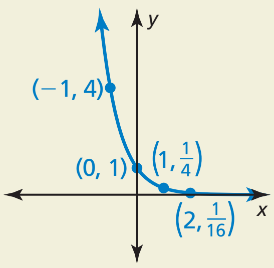
42. $r = 2.29\%$, compounded monthly

43. $r = 1.83\%$, compounded daily

44. $r = 1.26\%$, compounded monthly

45. **USING STRUCTURE** A website recorded the number y of referrals it received from social media websites over a 10-year period. The results can be modeled by $y = 2500(1.50)^t$, where t is the year and $0 \le t \le 9$. Interpret the values of a and b in this situation. What is the annual percent increase? Explain.

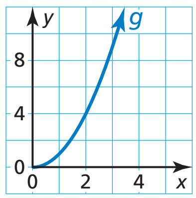
46. **HOW DO YOU SEE IT?** Consider the graph of an exponential function of the form $f(x) = ab^x$.



a. Determine whether the graph of f represents exponential growth or exponential decay.

b. What are the domain and range of the function? Explain.

47. **MAKING AN ARGUMENT** Your friend says the graph of $f(x) = 2^x$ increases at a faster rate than the graph of $g(x) = x^2$ when $x \ge 0$. Is your friend correct? Explain your reasoning.



48. **THOUGHT PROVOKING** The function $f(x) = b^x$ represents an exponential decay function. Write a second exponential decay function in terms of b and x.

49. **PROBLEM SOLVING** The population p of a small town after x years can be modeled by the function $p = 6850(1.03)^x$. What is the average rate of change in the population over the first 6 years? Justify your answer.

50. **REASONING** Consider the exponential function $f(x) = ab^x$.

a. Show that $\dfrac{f(x+1)}{f(x)} = b$.

b. Use the equation in part (a) to explain why there is no exponential function of the form $f(x) = ab^x$ whose graph passes through the points in the table below.

x	0	1	2	3	4
y	4	4	8	24	72

51. **PROBLEM SOLVING** The number E of eggs a Leghorn chicken produces per year can be modeled by the equation $E = 179.2(0.89)^{w/52}$, where w is the age (in weeks) of the chicken and $w \ge 22$.

a. Identify the decay factor and the percent decrease.

b. Graph the model.

c. Estimate the egg production of a chicken that is 2.5 years old.

d. Explain how you can rewrite the given equation so that time is measured in years rather than in weeks.

52. **CRITICAL THINKING** You buy a new stereo for \$1300 and are able to sell it 4 years later for \$275. Assume that the resale value of the stereo decays exponentially with time. Write an equation giving the resale value V (in dollars) of the stereo as a function of the time t (in years) since you bought it.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Simplify the expression. *(Skills Review Handbook)*

53. $x^9 \cdot x^2$

54. $\dfrac{x^4}{x^3}$

55. $4x \cdot 6x$

56. $\left(\dfrac{4x^8}{2x^6}\right)^4$

57. $\dfrac{x + 3x}{2}$

58. $\dfrac{6x}{2} + 4x$

59. $\dfrac{12x}{4x} + 5x$

60. $(2x \cdot 3x^5)^3$

6.2 The Natural Base e

Essential Question

What is the natural base e?

So far in your study of mathematics, you have worked with special numbers such as π and i. Another special number is called the *natural base* and is denoted by e. The natural base e is irrational, so you cannot find its exact value.

EXPLORATION 1 Approximating the Natural Base e

Work with a partner. One way to approximate the natural base e is to approximate the sum

$$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots.$$

Use a spreadsheet or a graphing calculator to approximate this sum. Explain the steps you used. How many decimal places did you use in your approximation?

USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technological tools to explore and deepen your understanding of concepts.

EXPLORATION 2 Approximating the Natural Base e

Work with a partner. Another way to approximate the natural base e is to consider the expression

$$\left(1 + \frac{1}{x}\right)^x.$$

As x increases, the value of this expression approaches the value of e. Copy and complete the table. Then use the results in the table to approximate e. Compare this approximation to the one you obtained in Exploration 1.

x	10^1	10^2	10^3	10^4	10^5	10^6
$\left(1 + \frac{1}{x}\right)^x$						

EXPLORATION 3 Graphing a Natural Base Function

Work with a partner. Use your approximate value of e in Exploration 1 or 2 to complete the table. Then sketch the graph of the *natural base exponential function* $y = e^x$. You can use a graphing calculator and the e^x key to check your graph. What are the domain and range of $y = e^x$? Justify your answers.

x	-2	-1	0	1	2
$y = e^x$					

Communicate Your Answer

4. What is the natural base e?

5. Repeat Exploration 3 for the natural base exponential function $y = e^{-x}$. Then compare the graph of $y = e^x$ to the graph of $y = e^{-x}$.

6. The natural base e is used in a wide variety of real-life applications. Use the Internet or some other reference to research some of the real-life applications of e.

6.2 Lesson

Core Vocabulary

natural base *e*, *p. 304*

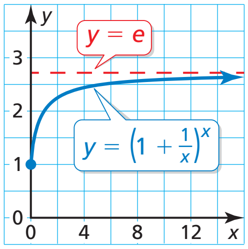
Previous
irrational number
properties of exponents
percent increase
percent decrease
compound interest

What You Will Learn

- Define and use the natural base e.
- Graph natural base functions.
- Solve real-life problems.

The Natural Base e

The history of mathematics is marked by the discovery of special numbers, such as π and i. Another special number is denoted by the letter e. The number is called the **natural base *e***. The expression $\left(1 + \frac{1}{x}\right)^x$ approaches e as x increases, as shown in the graph and table.



x	10^1	10^2	10^3	10^4	10^5	10^6
$\left(1 + \frac{1}{x}\right)^x$	2.59374	2.70481	2.71692	2.71815	2.71827	2.71828

Core Concept

The Natural Base e

The natural base e is irrational. It is defined as follows:

As x approaches $+\infty$, $\left(1 + \frac{1}{x}\right)^x$ approaches $e \approx 2.71828182846$.

EXAMPLE 1 Simplifying Natural Base Expressions

Simplify each expression.

a. $e^3 \cdot e^6$ **b.** $\frac{16e^5}{4e^4}$ **c.** $(3e^{-4x})^2$

SOLUTION

a. $e^3 \cdot e^6 = e^{3+6}$

$= e^9$

b. $\frac{16e^5}{4e^4} = 4e^{5-4}$

$= 4e$

c. $(3e^{-4x})^2 = 3^2(e^{-4x})^2$

$= 9e^{-8x}$

$= \frac{9}{e^{8x}}$

Check

You can use a calculator to check the equivalence of numerical expressions involving e.

```
e^(3)*e^(6)
     8103.083928
e^(9)
     8103.083928
```

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Simplify the expression.

1. $e^7 \cdot e^4$ **2.** $\frac{24e^8}{8e^5}$ **3.** $(10e^{-3x})^3$

Graphing Natural Base Functions

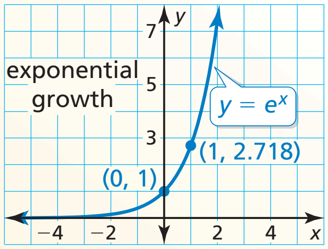
Core Concept

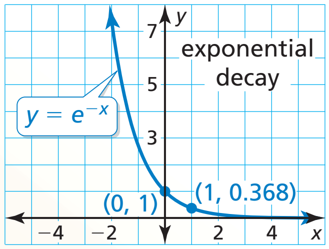
Natural Base Functions

A function of the form $y = ae^{rx}$ is called a *natural base exponential function*.

- When $a > 0$ and $r > 0$, the function is an exponential growth function.
- When $a > 0$ and $r < 0$, the function is an exponential decay function.

The graphs of the basic functions $y = e^x$ and $y = e^{-x}$ are shown.





EXAMPLE 2 Graphing Natural Base Functions

Tell whether each function represents *exponential growth* or *exponential decay*. Then graph the function.

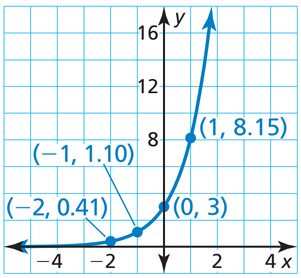
a. $y = 3e^x$

b. $f(x) = e^{-0.5x}$

SOLUTION

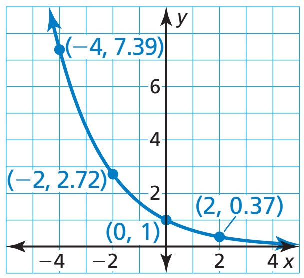
a. Because $a = 3$ is positive and $r = 1$ is positive, the function is an exponential growth function. Use a table to graph the function.

x	−2	−1	0	1
y	0.41	1.10	3	8.15



b. Because $a = 1$ is positive and $r = -0.5$ is negative, the function is an exponential decay function. Use a table to graph the function.

x	−4	−2	0	2
y	7.39	2.72	1	0.37



LOOKING FOR STRUCTURE

You can rewrite natural base exponential functions to find percent rates of change. In Example 2(b),

$$f(x) = e^{-0.5x} = (e^{-0.5})^x \approx (0.6065)^x = (1 - 0.3935)^x.$$

So, the percent decrease is about 39.35%.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

Tell whether the function represents *exponential growth* or *exponential decay*. Then graph the function.

4. $y = \frac{1}{2}e^x$

5. $y = 4e^{-x}$

6. $f(x) = 2e^{2x}$

Solving Real-Life Problems

You have learned that the balance of an account earning compound interest is given by $A = P\left(1 + \frac{r}{n}\right)^{nt}$. As the frequency n of compounding approaches positive infinity, the compound interest formula approximates the following formula.

Core Concept

Continuously Compounded Interest

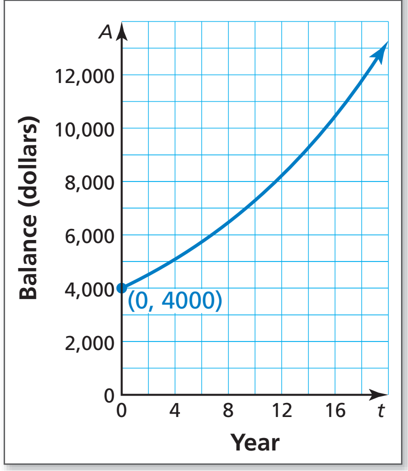
When interest is compounded *continuously*, the amount A in an account after t years is given by the formula

$$A = Pe^{rt}$$

where P is the principal and r is the annual interest rate expressed as a decimal.

EXAMPLE 3 Modeling with Mathematics

Your Friend's Account



You and your friend each have accounts that earn annual interest compounded continuously. The balance A (in dollars) of your account after t years can be modeled by $A = 4500e^{0.04t}$. The graph shows the balance of your friend's account over time. Which account has a greater principal? Which has a greater balance after 10 years?

SOLUTION

1. **Understand the Problem** You are given a graph and an equation that represent account balances. You are asked to identify the account with the greater principal and the account with the greater balance after 10 years.

2. **Make a Plan** Use the equation to find your principal and account balance after 10 years. Then compare these values to the graph of your friend's account.

3. **Solve the Problem** The equation $A = 4500e^{0.04t}$ is of the form $A = Pe^{rt}$, where $P = 4500$. So, your principal is \$4500. Your balance A when $t = 10$ is

$$A = 4500e^{0.04(10)} = \$6713.21.$$

Because the graph passes through (0, 4000), your friend's principal is \$4000. The graph also shows that the balance is about \$7250 when $t = 10$.

▶ So, your account has a greater principal, but your friend's account has a greater balance after 10 years.

4. **Look Back** Because your friend's account has a lesser principal but a greater balance after 10 years, the average rate of change from $t = 0$ to $t = 10$ should be greater for your friend's account than for your account.

Your account: $\frac{A(10) - A(0)}{10 - 0} = \frac{6713.21 - 4500}{10} = 221.321$

Your friend's account: $\frac{A(10) - A(0)}{10 - 0} \approx \frac{7250 - 4000}{10} = 325$ ✓

MAKING CONJECTURES

You can also use this reasoning to conclude that your friend's account has a greater annual interest rate than your account.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

7. You deposit \$4250 in an account that earns 5% annual interest compounded continuously. Compare the balance after 10 years with the accounts in Example 3.

6.2 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **VOCABULARY** What is the natural base e?

2. **WRITING** Tell whether the function $f(x) = \frac{1}{3}e^{4x}$ represents exponential growth or exponential decay. Explain.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–12, simplify the expression. *(See Example 1.)*

3. $e^3 \cdot e^5$
4. $e^{-4} \cdot e^6$
5. $\dfrac{11e^9}{22e^{10}}$
6. $\dfrac{27e^7}{3e^4}$
7. $(5e^{7x})^4$
8. $(4e^{-2x})^3$
9. $\sqrt{9e^{6x}}$
10. $\sqrt[3]{8e^{12x}}$
11. $e^x \cdot e^{-6x} \cdot e^8$
12. $e^x \cdot e^4 \cdot e^{x+3}$

ERROR ANALYSIS **In Exercises 13 and 14, describe and correct the error in simplifying the expression.**

13. ✗
$$(4e^{3x})^2 = 4e^{(3x)(2)}$$
$$= 4e^{6x}$$

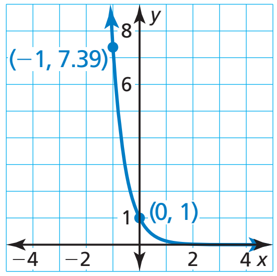
14. ✗
$$\frac{e^{5x}}{e^{-2x}} = e^{5x-2x}$$
$$= e^{3x}$$

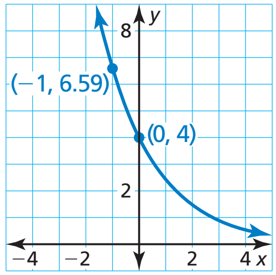
In Exercises 15–22, tell whether the function represents *exponential growth* or *exponential decay*. Then graph the function. *(See Example 2.)*

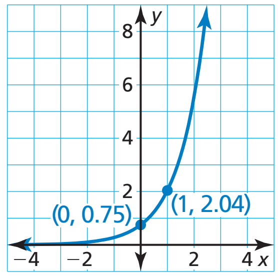
15. $y = e^{3x}$
16. $y = e^{-2x}$
17. $y = 2e^{-x}$
18. $y = 3e^{2x}$
19. $y = 0.5e^x$
20. $y = 0.25e^{-3x}$
21. $y = 0.4e^{-0.25x}$
22. $y = 0.6e^{0.5x}$

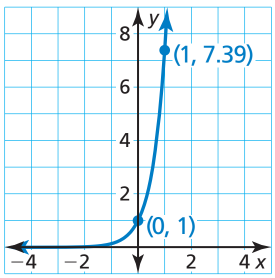
ANALYZING EQUATIONS **In Exercises 23–26, match the function with its graph. Explain your reasoning.**

23. $y = e^{2x}$
24. $y = e^{-2x}$
25. $y = 4e^{-0.5x}$
26. $y = 0.75e^x$

A.


B.


C.


D.


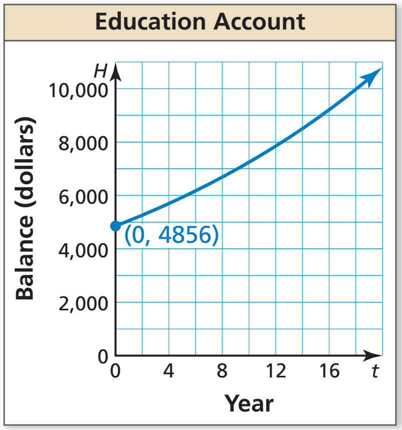
USING STRUCTURE **In Exercises 27–30, use the properties of exponents to rewrite the function in the form $y = a(1 + r)^t$ or $y = a(1 - r)^t$. Then find the percent rate of change.**

27. $y = e^{-0.25t}$
28. $y = e^{-0.75t}$
29. $y = 2e^{0.4t}$
30. $y = 0.5e^{0.8t}$

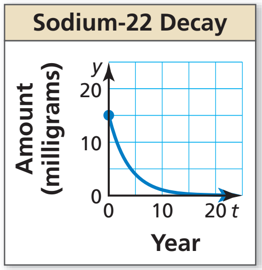
USING TOOLS **In Exercises 31–34, use a table of values or a graphing calculator to graph the function. Then identify the domain and range.**

31. $y = e^{x-2}$
32. $y = e^{x+1}$
33. $y = 2e^x + 1$
34. $y = 3e^x - 5$

35. MODELING WITH MATHEMATICS Investment accounts for a house and education earn annual interest compounded continuously. The balance H (in dollars) of the house fund after t years can be modeled by $H = 3224e^{0.05t}$. The graph shows the balance in the education fund over time. Which account has the greater principal? Which account has a greater balance after 10 years? *(See Example 3.)*



36. MODELING WITH MATHEMATICS Tritium and sodium-22 decay over time. In a sample of tritium, the amount y (in milligrams) remaining after t years is given by $y = 10e^{-0.0562t}$. The graph shows the amount of sodium-22 in a sample over time. Which sample started with a greater amount? Which has a greater amount after 10 years?



37. OPEN-ENDED Find values of a, b, r, and q such that $f(x) = ae^{rx}$ and $g(x) = be^{qx}$ are exponential decay functions, but $\frac{f(x)}{g(x)}$ represents exponential growth.

38. THOUGHT PROVOKING Explain why $A = P\left(1 + \frac{r}{n}\right)^{nt}$ approximates $A = Pe^{rt}$ as n approaches positive infinity.

39. WRITING Can the natural base e be written as a ratio of two integers? Explain.

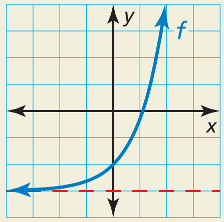
40. MAKING AN ARGUMENT Your friend evaluates $f(x) = e^{-x}$ when $x = 1000$ and concludes that the graph of $y = f(x)$ has an x-intercept at (1000, 0). Is your friend correct? Explain your reasoning.

41. DRAWING CONCLUSIONS You invest \$2500 in an account to save for college. Account 1 pays 6% annual interest compounded quarterly. Account 2 pays 4% annual interest compounded continuously. Which account should you choose to obtain the greater amount in 10 years? Justify your answer.

42. HOW DO YOU SEE IT? Use the graph to complete each statement.

a. $f(x)$ approaches ____ as x approaches $+\infty$.

b. $f(x)$ approaches ____ as x approaches $-\infty$.



43. PROBLEM SOLVING The growth of *Mycobacterium tuberculosis* bacteria can be modeled by the function $N(t) = ae^{0.166t}$, where N is the number of cells after t hours and a is the number of cells when $t = 0$.

a. At 1:00 P.M., there are 30 *M. tuberculosis* bacteria in a sample. Write a function that gives the number of bacteria after 1:00 P.M.

b. Use a graphing calculator to graph the function in part (a).

c. Describe how to find the number of cells in the sample at 3:45 P.M.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write the number in scientific notation. *(Skills Review Handbook)*

44. 0.006 **45.** 5000 **46.** 26,000,000 **47.** 0.000000047

Find the inverse of the function. Then graph the function and its inverse. *(Section 5.6)*

48. $y = 3x + 5$

49. $y = x^2 - 1, x \leq 0$

50. $y = \sqrt{x + 6}$

51. $y = x^3 - 2$

6.3 Logarithms and Logarithmic Functions

Essential Question What are some of the characteristics of the graph of a logarithmic function?

Every exponential function of the form $f(x) = b^x$, where b is a positive real number other than 1, has an inverse function that you can denote by $g(x) = \log_b x$. This inverse function is called a *logarithmic function with base b*.

EXPLORATION 1 Rewriting Exponential Equations

Work with a partner. Find the value of x in each exponential equation. Explain your reasoning. Then use the value of x to rewrite the exponential equation in its equivalent logarithmic form, $x = \log_b y$.

a. $2^x = 8$ **b.** $3^x = 9$ **c.** $4^x = 2$

d. $5^x = 1$ **e.** $5^x = \frac{1}{5}$ **f.** $8^x = 4$

EXPLORATION 2 Graphing Exponential and Logarithmic Functions

Work with a partner. Complete each table for the given exponential function. Use the results to complete the table for the given logarithmic function. Explain your reasoning. Then sketch the graphs of f and g in the same coordinate plane.

a.

x	−2	−1	0	1	2
$f(x) = 2^x$					

x					
$g(x) = \log_2 x$	−2	−1	0	1	2

b.

x	−2	−1	0	1	2
$f(x) = 10^x$					

x					
$g(x) = \log_{10} x$	−2	−1	0	1	2

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to justify your conclusions and communicate them to others.

EXPLORATION 3 Characteristics of Graphs of Logarithmic Functions

Work with a partner. Use the graphs you sketched in Exploration 2 to determine the domain, range, x-intercept, and asymptote of the graph of $g(x) = \log_b x$, where b is a positive real number other than 1. Explain your reasoning.

Communicate Your Answer

4. What are some of the characteristics of the graph of a logarithmic function?

5. How can you use the graph of an exponential function to obtain the graph of a logarithmic function?

6.3 Lesson

Core Vocabulary

logarithm of y with base b, *p. 310*
common logarithm, *p. 311*
natural logarithm, *p. 311*

Previous
inverse functions

What You Will Learn

- Define and evaluate logarithms.
- Use inverse properties of logarithmic and exponential functions.
- Graph logarithmic functions.

Logarithms

You know that $2^2 = 4$ and $2^3 = 8$. However, for what value of x does $2^x = 6$? Mathematicians define this x-value using a *logarithm* and write $x = \log_2 6$. The definition of a logarithm can be generalized as follows.

Core Concept

Definition of Logarithm with Base *b*

Let b and y be positive real numbers with $b \neq 1$. The **logarithm of *y* with base *b*** is denoted by $\log_b y$ and is defined as

$$\log_b y = x \qquad \text{if and only if} \qquad b^x = y.$$

The expression $\log_b y$ is read as "log base b of y."

This definition tells you that the equations $\log_b y = x$ and $b^x = y$ are equivalent. The first is in *logarithmic form*, and the second is in *exponential form*.

EXAMPLE 1 Rewriting Logarithmic Equations

Rewrite each equation in exponential form.

a. $\log_2 16 = 4$ **b.** $\log_4 1 = 0$ **c.** $\log_{12} 12 = 1$ **d.** $\log_{1/4} 4 = -1$

SOLUTION

	Logarithmic Form	Exponential Form
a.	$\log_2 16 = 4$	$2^4 = 16$
b.	$\log_4 1 = 0$	$4^0 = 1$
c.	$\log_{12} 12 = 1$	$12^1 = 12$
d.	$\log_{1/4} 4 = -1$	$\left(\frac{1}{4}\right)^{-1} = 4$

EXAMPLE 2 Rewriting Exponential Equations

Rewrite each equation in logarithmic form.

a. $5^2 = 25$ **b.** $10^{-1} = 0.1$ **c.** $8^{2/3} = 4$ **d.** $6^{-3} = \frac{1}{216}$

SOLUTION

	Exponential Form	Logarithmic Form
a.	$5^2 = 25$	$\log_5 25 = 2$
b.	$10^{-1} = 0.1$	$\log_{10} 0.1 = -1$
c.	$8^{2/3} = 4$	$\log_8 4 = \frac{2}{3}$
d.	$6^{-3} = \frac{1}{216}$	$\log_6 \frac{1}{216} = -3$

Parts (b) and (c) of Example 1 illustrate two special logarithm values that you should learn to recognize. Let b be a positive real number such that $b \neq 1$.

Logarithm of 1	**Logarithm of *b* with Base *b***
$\log_b 1 = 0$ because $b^0 = 1$.	$\log_b b = 1$ because $b^1 = b$.

EXAMPLE 3 Evaluating Logarithmic Expressions

Evaluate each logarithm.

a. $\log_4 64$ **b.** $\log_5 0.2$ **c.** $\log_{1/5} 125$ **d.** $\log_{36} 6$

SOLUTION

To help you find the value of $\log_b y$, ask yourself what power of b gives you y.

a. What power of 4 gives you 64? $4^3 = 64$, so $\log_4 64 = 3$.

b. What power of 5 gives you 0.2? $5^{-1} = 0.2$, so $\log_5 0.2 = -1$.

c. What power of $\frac{1}{5}$ gives you 125? $\left(\frac{1}{5}\right)^{-3} = 125$, so $\log_{1/5} 125 = -3$.

d. What power of 36 gives you 6? $36^{1/2} = 6$, so $\log_{36} 6 = \frac{1}{2}$.

A **common logarithm** is a logarithm with base 10. It is denoted by $\log_{10}$ or simply by log. A **natural logarithm** is a logarithm with base e. It can be denoted by $\log_e$ but is usually denoted by ln.

Common Logarithm	**Natural Logarithm**
$\log_{10} x = \log x$	$\log_e x = \ln x$

EXAMPLE 4 Evaluating Common and Natural Logarithms

Evaluate (a) log 8 and (b) ln 0.3 using a calculator. Round your answer to three decimal places.

SOLUTION

Most calculators have keys for evaluating common and natural logarithms.

```
log(8)
      .903089987
ln(0.3)
    -1.203972804
```

a. $\log 8 \approx 0.903$

b. $\ln 0.3 \approx -1.204$

Check your answers by rewriting each logarithm in exponential form and evaluating.

Check

```
10^(0.903)
       7.99834255
e^(-1.204)
      .2999918414
```

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Rewrite the equation in exponential form.

1. $\log_3 81 = 4$ **2.** $\log_7 7 = 1$ **3.** $\log_{14} 1 = 0$ **4.** $\log_{1/2} 32 = -5$

Rewrite the equation in logarithmic form.

5. $7^2 = 49$ **6.** $50^0 = 1$ **7.** $4^{-1} = \frac{1}{4}$ **8.** $256^{1/8} = 2$

Evaluate the logarithm. If necessary, use a calculator and round your answer to three decimal places.

9. $\log_2 32$ **10.** $\log_{27} 3$ **11.** $\log 12$ **12.** $\ln 0.75$

Using Inverse Properties

By the definition of a logarithm, it follows that the logarithmic function $g(x) = \log_b x$ is the inverse of the exponential function $f(x) = b^x$. This means that

$$g(f(x)) = \log_b b^x = x \qquad \text{and} \qquad f(g(x)) = b^{\log_b x} = x.$$

In other words, exponential functions and logarithmic functions "undo" each other.

EXAMPLE 5 Using Inverse Properties

Simplify (a) $10^{\log 4}$ and (b) $\log_5 25^x$.

SOLUTION

a. $10^{\log 4} = 4$ — $b^{\log_b x} = x$

b. $\log_5 25^x = \log_5(5^2)^x$ — Express 25 as a power with base 5.

$= \log_5 5^{2x}$ — Power of a Power Property

$= 2x$ — $\log_b b^x = x$

EXAMPLE 6 Finding Inverse Functions

Find the inverse of each function.

a. $f(x) = 6^x$ **b.** $y = \ln(x + 3)$

SOLUTION

a. From the definition of logarithm, the inverse of $f(x) = 6^x$ is $g(x) = \log_6 x$.

b. $y = \ln(x + 3)$ — Write original function.

$x = \ln(y + 3)$ — Switch x and y.

$e^x = y + 3$ — Write in exponential form.

$e^x - 3 = y$ — Subtract 3 from each side.

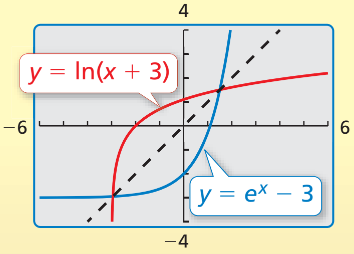
▶ The inverse of $y = \ln(x + 3)$ is $y = e^x - 3$.

Check

a. $f(g(x)) = 6^{\log_6 x} = x$ ✓

$g(f(x)) = \log_6 6^x = x$ ✓

b.



The graphs appear to be reflections of each other in the line $y = x$. ✓

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Simplify the expression.

13. $8^{\log_8 x}$ **14.** $\log_7 7^{-3x}$ **15.** $\log_2 64^x$ **16.** $e^{\ln 20}$

17. Find the inverse of $y = 4^x$. **18.** Find the inverse of $y = \ln(x - 5)$.

Graphing Logarithmic Functions

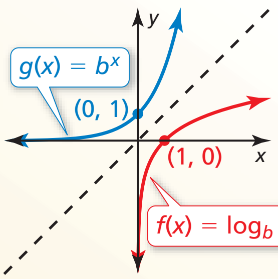
You can use the inverse relationship between exponential and logarithmic functions to graph logarithmic functions.

Core Concept

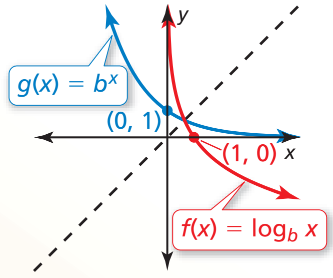
Parent Graphs for Logarithmic Functions

The graph of $f(x) = \log_b x$ is shown below for $b > 1$ and for $0 < b < 1$. Because $f(x) = \log_b x$ and $g(x) = b^x$ are inverse functions, the graph of $f(x) = \log_b x$ is the reflection of the graph of $g(x) = b^x$ in the line $y = x$.

Graph of $f(x) = \log_b x$ for $b > 1$



Graph of $f(x) = \log_b x$ for $0 < b < 1$



Note that the y-axis is a vertical asymptote of the graph of $f(x) = \log_b x$. The domain of $f(x) = \log_b x$ is $x > 0$, and the range is all real numbers.

EXAMPLE 7 Graphing a Logarithmic Function

Graph $f(x) = \log_3 x$.

SOLUTION

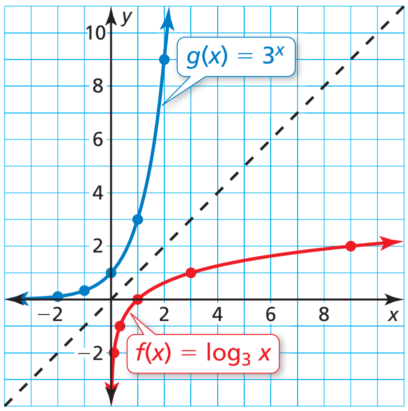
Step 1 Find the inverse of f. From the definition of logarithm, the inverse of $f(x) = \log_3 x$ is $g(x) = 3^x$.

Step 2 Make a table of values for $g(x) = 3^x$.

x	-2	-1	0	1	2
$g(x)$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

Step 3 Plot the points from the table and connect them with a smooth curve.

Step 4 Because $f(x) = \log_3 x$ and $g(x) = 3^x$ are inverse functions, the graph of f is obtained by reflecting the graph of g in the line $y = x$. To do this, reverse the coordinates of the points on g and plot these new points on the graph of f.



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Graph the function.

19. $y = \log_2 x$ **20.** $f(x) = \log_5 x$ **21.** $y = \log_{1/2} x$

6.3 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** A logarithm with base 10 is called a(n) ___________ logarithm.

2. **COMPLETE THE SENTENCE** The expression $\log_3 9$ is read as ______________.

3. **WRITING** Describe the relationship between $y = 7^x$ and $y = \log_7 x$.

4. **DIFFERENT WORDS, SAME QUESTION** Which is different? Find "both" answers.

What power of 4 gives you 16?	What is log base 4 of 16?
Evaluate 4^2.	Evaluate $\log_4 16$.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–10, rewrite the equation in exponential form. *(See Example 1.)*

5. $\log_3 9 = 2$
6. $\log_4 4 = 1$
7. $\log_6 1 = 0$
8. $\log_7 343 = 3$
9. $\log_{1/2} 16 = -4$
10. $\log_3 \frac{1}{3} = -1$

In Exercises 11–16, rewrite the equation in logarithmic form. *(See Example 2.)*

11. $6^2 = 36$
12. $12^0 = 1$
13. $16^{-1} = \frac{1}{16}$
14. $5^{-2} = \frac{1}{25}$
15. $125^{2/3} = 25$
16. $49^{1/2} = 7$

In Exercises 17–24, evaluate the logarithm. *(See Example 3.)*

17. $\log_3 81$
18. $\log_7 49$
19. $\log_3 3$
20. $\log_{1/2} 1$
21. $\log_5 \frac{1}{625}$
22. $\log_8 \frac{1}{512}$
23. $\log_4 0.25$
24. $\log_{10} 0.001$

25. **NUMBER SENSE** Order the logarithms from least value to greatest value.

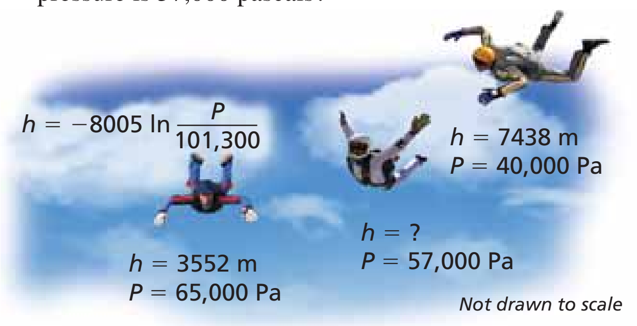
$\log_5 23$	$\log_6 38$	$\log_7 8$	$\log_2 10$

26. **WRITING** Explain why the expressions $\log_2(-1)$ and $\log_1 1$ are not defined.

In Exercises 27–32, evaluate the logarithm using a calculator. Round your answer to three decimal places. *(See Example 4.)*

27. $\log 6$
28. $\ln 12$
29. $\ln \frac{1}{3}$
30. $\log \frac{2}{7}$
31. $3 \ln 0.5$
32. $\log 0.6 + 1$

33. **MODELING WITH MATHEMATICS** Skydivers use an instrument called an *altimeter* to track their altitude as they fall. The altimeter determines altitude by measuring air pressure. The altitude h (in meters) above sea level is related to the air pressure P (in pascals) by the function shown in the diagram. What is the altitude above sea level when the air pressure is 57,000 pascals?



34. **MODELING WITH MATHEMATICS** The pH value for a substance measures how acidic or alkaline the substance is. It is given by the formula $\text{pH} = -\log[H^+]$, where H^+ is the hydrogen ion concentration (in moles per liter). Find the pH of each substance.

    a. baking soda: $[H^+] = 10^{-8}$ moles per liter

    b. vinegar: $[H^+] = 10^{-3}$ moles per liter

In Exercises 35–40, simplify the expression. *(See Example 5.)*

35. $7^{\log_7 x}$ **36.** $3^{\log_3 5x}$

37. $e^{\ln 4}$ **38.** $10^{\log 15}$

39. $\log_3 3^{2x}$ **40.** $\ln e^{x+1}$

41. ERROR ANALYSIS Describe and correct the error in rewriting $4^{-3} = \frac{1}{64}$ in logarithmic form.

✗ $\log_4(-3) = \frac{1}{64}$

42. ERROR ANALYSIS Describe and correct the error in simplifying the expression $\log_4 64^x$.

✗
$$\log_4 64^x = \log_4(16 \cdot 4^x)$$
$$= \log_4(4^2 \cdot 4^x)$$
$$= \log_4 4^{2+x}$$
$$= 2 + x$$

In Exercises 43–52, find the inverse of the function. *(See Example 6.)*

43. $y = 0.3^x$ **44.** $y = 11^x$

45. $y = \log_2 x$ **46.** $y = \log_{1/5} x$

47. $y = \ln(x - 1)$ **48.** $y = \ln 2x$

49. $y = e^{3x}$ **50.** $y = e^{x-4}$

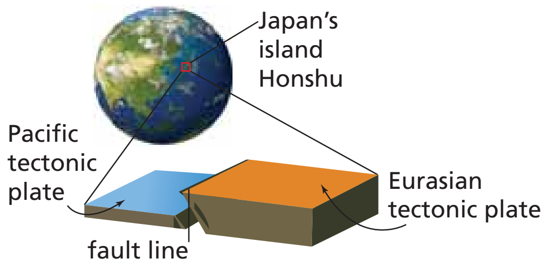
51. $y = 5^x - 9$ **52.** $y = 13 + \log x$

53. PROBLEM SOLVING The wind speed s (in miles per hour) near the center of a tornado can be modeled by $s = 93 \log d + 65$, where d is the distance (in miles) that the tornado travels.

a. In 1925, a tornado traveled 220 miles through three states. Estimate the wind speed near the center of the tornado.

b. Find the inverse of the given function. Describe what the inverse represents.

54. MODELING WITH MATHEMATICS The energy magnitude M of an earthquake can be modeled by $M = \frac{2}{3} \log E - 9.9$, where E is the amount of energy released (in ergs).



a. In 2011, a powerful earthquake in Japan, caused by the slippage of two tectonic plates along a fault, released 2.24×10^{28} ergs. What was the energy magnitude of the earthquake?

b. Find the inverse of the given function. Describe what the inverse represents.

In Exercises 55–60, graph the function. *(See Example 7.)*

55. $y = \log_4 x$ **56.** $y = \log_6 x$

57. $y = \log_{1/3} x$ **58.** $y = \log_{1/4} x$

59. $y = \log_2 x - 1$ **60.** $y = \log_3(x + 2)$

USING TOOLS In Exercises 61–64, use a graphing calculator to graph the function. Determine the domain, range, and asymptote of the function.

61. $y = \log(x + 2)$ **62.** $y = -\ln x$

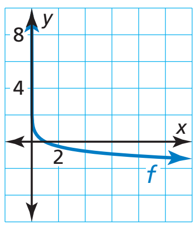
63. $y = \ln(-x)$ **64.** $y = 3 - \log x$

65. MAKING AN ARGUMENT Your friend states that every logarithmic function will pass through the point (1, 0). Is your friend correct? Explain your reasoning.

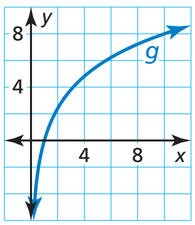
66. ANALYZING RELATIONSHIPS Rank the functions in order from the least average rate of change to the greatest average rate of change over the interval $1 \leq x \leq 10$.

a. $y = \log_6 x$ **b.** $y = \log_{3/5} x$

c.



d.



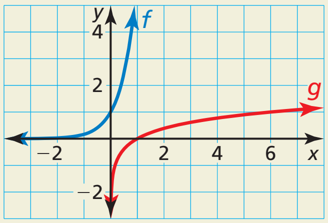
67. PROBLEM SOLVING Biologists have found that the length ℓ (in inches) of an alligator and its weight w (in pounds) are related by the function $\ell = 27.1 \ln w - 32.8$.

a. Use a graphing calculator to graph the function.

b. Use your graph to estimate the weight of an alligator that is 10 feet long.

c. Use the *zero* feature to find the x-intercept of the graph of the function. Does this x-value make sense in the context of the situation? Explain.

68. HOW DO YOU SEE IT? The figure shows the graphs of the two functions f and g.



a. Compare the end behavior of the logarithmic function g to that of the exponential function f.

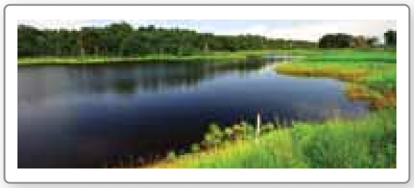
b. Determine whether the functions are inverse functions. Explain.

c. What is the base of each function? Explain.

69. PROBLEM SOLVING A study in Florida found that the number s of fish species in a pool or lake can be modeled by the function

$$s = 30.6 - 20.5 \log A + 3.8(\log A)^2$$

where A is the area (in square meters) of the pool or lake.



a. Use a graphing calculator to graph the function on the domain $200 \le A \le 35{,}000$.

b. Use your graph to estimate the number of species in a lake with an area of 30,000 square meters.

c. Use your graph to estimate the area of a lake that contains six species of fish.

d. Describe what happens to the number of fish species as the area of a pool or lake increases. Explain why your answer makes sense.

70. THOUGHT PROVOKING Write a logarithmic function that has an output of −4. Then sketch the graph of your function.

71. CRITICAL THINKING Evaluate each logarithm. (*Hint*: For each logarithm $\log_b x$, rewrite b and x as powers of the same base.)

a. $\log_{125} 25$ **b.** $\log_8 32$

c. $\log_{27} 81$ **d.** $\log_4 128$

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Let $f(x) = \sqrt[3]{x}$. Write a rule for g that represents the indicated transformation of the graph of f. *(Section 5.3)*

72. $g(x) = -f(x)$

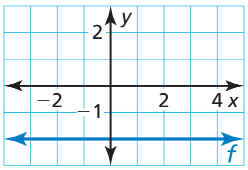
73. $g(x) = f\left(\frac{1}{2}x\right)$

74. $g(x) = f(-x) + 3$

75. $g(x) = f(x + 2)$

Identify the function family to which f belongs. Compare the graph of f to the graph of its parent function. *(Section 1.1)*

76.



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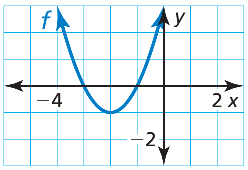

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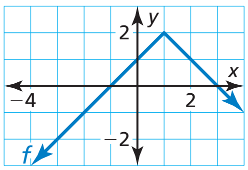


6.4 Transformations of Exponential and Logarithmic Functions

Essential Question

How can you transform the graphs of exponential and logarithmic functions?

EXPLORATION 1 Identifying Transformations

Work with a partner. Each graph shown is a transformation of the parent function

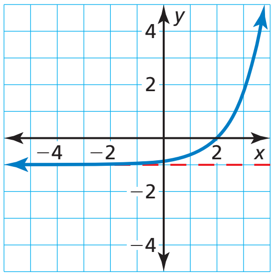
$f(x) = e^x$ or $f(x) = \ln x$.

Match each function with its graph. Explain your reasoning. Then describe the transformation of f represented by g.

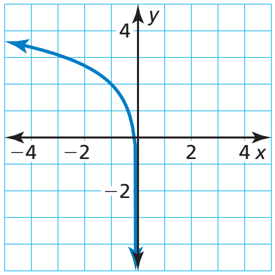
a. $g(x) = e^{x+2} - 3$ **b.** $g(x) = -e^{x+2} + 1$ **c.** $g(x) = e^{x-2} - 1$

d. $g(x) = \ln(x + 2)$ **e.** $g(x) = 2 + \ln x$ **f.** $g(x) = 2 + \ln(-x)$

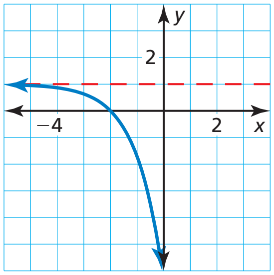
A.



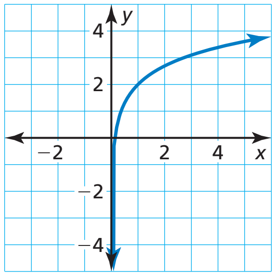
B.



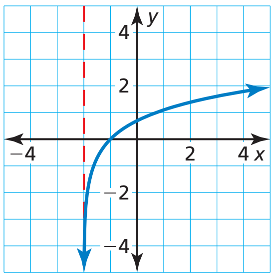
C.



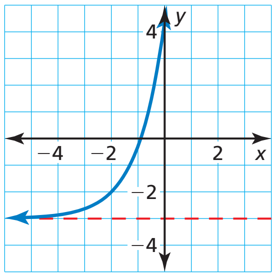
D.



E.



F.



EXPLORATION 2 Characteristics of Graphs

Work with a partner. Determine the domain, range, and asymptote of each function in Exploration 1. Justify your answers.

REASONING QUANTITATIVELY

To be proficient in math, you need to make sense of quantities and their relationships in problem situations.

Communicate Your Answer

3. How can you transform the graphs of exponential and logarithmic functions?

4. Find the inverse of each function in Exploration 1. Then check your answer by using a graphing calculator to graph each function and its inverse in the same viewing window.

6.4 Lesson

Core Vocabulary

Previous
exponential function
logarithmic function
transformations

What You Will Learn

- Transform graphs of exponential functions.
- Transform graphs of logarithmic functions.
- Write transformations of graphs of exponential and logarithmic functions.

Transforming Graphs of Exponential Functions

You can transform graphs of exponential and logarithmic functions in the same way you transformed graphs of functions in previous chapters. Examples of transformations of the graph of $f(x) = 4^x$ are shown below.

Core Concept

Transformation	$f(x)$ Notation	Examples	
Horizontal Translation Graph shifts left or right.	$f(x - h)$	$g(x) = 4^{x-3}$ $g(x) = 4^{x+2}$	3 units right 2 units left
Vertical Translation Graph shifts up or down.	$f(x) + k$	$g(x) = 4^x + 5$ $g(x) = 4^x - 1$	5 units up 1 unit down
Reflection Graph flips over x- or y-axis.	$f(-x)$ $-f(x)$	$g(x) = 4^{-x}$ $g(x) = -4^x$	in the y-axis in the x-axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y-axis.	$f(ax)$	$g(x) = 4^{2x}$ $g(x) = 4^{x/2}$	shrink by a factor of $\frac{1}{2}$ stretch by a factor of 2
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x-axis.	$a \cdot f(x)$	$g(x) = 3(4^x)$ $g(x) = \frac{1}{4}(4^x)$	stretch by a factor of 3 shrink by a factor of $\frac{1}{4}$

EXAMPLE 1 Translating an Exponential Function

Describe the transformation of $f(x) = \left(\frac{1}{2}\right)^x$ represented by $g(x) = \left(\frac{1}{2}\right)^x - 4$. Then graph each function.

SOLUTION

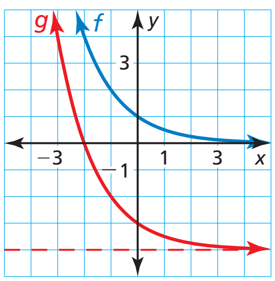
Notice that the function is of the form $g(x) = \left(\frac{1}{2}\right)^x + k$.

Rewrite the function to identify k.

$$g(x) = \left(\frac{1}{2}\right)^x + (-4)$$

k

▶ Because $k = -4$, the graph of g is a translation 4 units down of the graph of f.



STUDY TIP

Notice in the graph that the vertical translation also shifted the asymptote 4 units down, so the range of g is $y > -4$.

EXAMPLE 2 Translating a Natural Base Exponential Function

Describe the transformation of $f(x) = e^x$ represented by $g(x) = e^{x+3} + 2$. Then graph each function.

SOLUTION

STUDY TIP

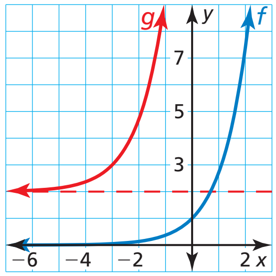
Notice in the graph that the vertical translation also shifted the asymptote 2 units up, so the range of g is $y > 2$.

Notice that the function is of the form $g(x) = e^{x-h} + k$. Rewrite the function to identify h and k.

$$g(x) = e^{x-(-3)} + 2$$

($h = -3$, $k = 2$)

▶ Because $h = -3$ and $k = 2$, the graph of g is a translation 3 units left and 2 units up of the graph of f.



EXAMPLE 3 Transforming Exponential Functions

Describe the transformation of f represented by g. Then graph each function.

a. $f(x) = 3^x$, $g(x) = 3^{3x-5}$

b. $f(x) = e^{-x}$, $g(x) = -\frac{1}{8}e^{-x}$

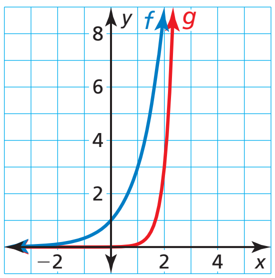
LOOKING FOR STRUCTURE

In Example 3(a), the horizontal shrink follows the translation. In the function $h(x) = 3^{3(x-5)}$, the translation 5 units right follows the horizontal shrink by a factor of $\frac{1}{3}$.

SOLUTION

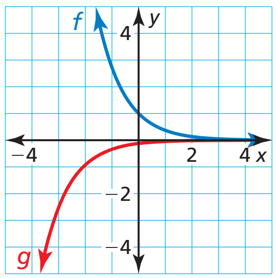
a. Notice that the function is of the form $g(x) = 3^{ax-h}$, where $a = 3$ and $h = 5$.

▶ So, the graph of g is a translation 5 units right, followed by a horizontal shrink by a factor of $\frac{1}{3}$ of the graph of f.



b. Notice that the function is of the form $g(x) = ae^{-x}$, where $a = -\frac{1}{8}$.

▶ So, the graph of g is a reflection in the x-axis and a vertical shrink by a factor of $\frac{1}{8}$ of the graph of f.



Monitoring Progress

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Describe the transformation of f represented by g. Then graph each function.

1. $f(x) = 2^x$, $g(x) = 2^{x-3} + 1$

2. $f(x) = e^{-x}$, $g(x) = e^{-x} - 5$

3. $f(x) = 0.4^x$, $g(x) = 0.4^{-2x}$

4. $f(x) = e^x$, $g(x) = -e^{x+6}$

Transforming Graphs of Logarithmic Functions

Examples of transformations of the graph of $f(x) = \log x$ are shown below.

Core Concept

Transformation	$f(x)$ Notation	Examples	
Horizontal Translation Graph shifts left or right.	$f(x - h)$	$g(x) = \log(x - 4)$ $g(x) = \log(x + 7)$	4 units right 7 units left
Vertical Translation Graph shifts up or down.	$f(x) + k$	$g(x) = \log x + 3$ $g(x) = \log x - 1$	3 units up 1 unit down
Reflection Graph flips over x- or y-axis.	$f(-x)$ $-f(x)$	$g(x) = \log(-x)$ $g(x) = -\log x$	in the y-axis in the x-axis
Horizontal Stretch or Shrink Graph stretches away from or shrinks toward y-axis.	$f(ax)$	$g(x) = \log(4x)$ $g(x) = \log\left(\frac{1}{3}x\right)$	shrink by a factor of $\frac{1}{4}$ stretch by a factor of 3
Vertical Stretch or Shrink Graph stretches away from or shrinks toward x-axis.	$a \cdot f(x)$	$g(x) = 5 \log x$ $g(x) = \frac{2}{3} \log x$	stretch by a factor of 5 shrink by a factor of $\frac{2}{3}$

EXAMPLE 4 Transforming Logarithmic Functions

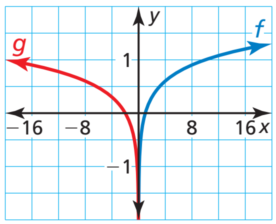
Describe the transformation of f represented by g. Then graph each function.

a. $f(x) = \log x,\ g(x) = \log\left(-\frac{1}{2}x\right)$ **b.** $f(x) = \log_{1/2} x,\ g(x) = 2 \log_{1/2}(x + 4)$

SOLUTION

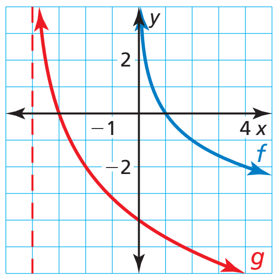
a. Notice that the function is of the form $g(x) = \log(ax)$, where $a = -\frac{1}{2}$.

► So, the graph of g is a reflection in the y-axis and a horizontal stretch by a factor of 2 of the graph of f.



b. Notice that the function is of the form $g(x) = a \log_{1/2}(x - h)$, where $a = 2$ and $h = -4$.

► So, the graph of g is a horizontal translation 4 units left and a vertical stretch by a factor of 2 of the graph of f.



STUDY TIP

In Example 4(b), notice in the graph that the horizontal translation also shifted the asymptote 4 units left, so the domain of g is $x > -4$.

Monitoring Progress

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Describe the transformation of *f* represented by *g*. Then graph each function.

5. $f(x) = \log_2 x$, $g(x) = -3 \log_2 x$

6. $f(x) = \log_{1/4} x$, $g(x) = \log_{1/4}(4x) - 5$

Writing Transformations of Graphs of Functions

EXAMPLE 5 Writing a Transformed Exponential Function

Let the graph of g be a reflection in the x-axis followed by a translation 4 units right of the graph of $f(x) = 2^x$. Write a rule for g.

SOLUTION

Step 1 First write a function h that represents the reflection of f.

$h(x) = -f(x)$ Multiply the output by −1.

$= -2^x$ Substitute 2^x for $f(x)$.

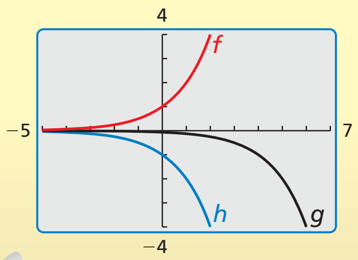
Step 2 Then write a function g that represents the translation of h.

$g(x) = h(x - 4)$ Subtract 4 from the input.

$= -2^{x-4}$ Replace x with $x - 4$ in $h(x)$.

▶ The transformed function is $g(x) = -2^{x-4}$.

Check



EXAMPLE 6 Writing a Transformed Logarithmic Function

Let the graph of g be a translation 2 units up followed by a vertical stretch by a factor of 2 of the graph of $f(x) = \log_{1/3} x$. Write a rule for g.

SOLUTION

Step 1 First write a function h that represents the translation of f.

$h(x) = f(x) + 2$ Add 2 to the output.

$= \log_{1/3} x + 2$ Substitute $\log_{1/3} x$ for $f(x)$.

Step 2 Then write a function g that represents the vertical stretch of h.

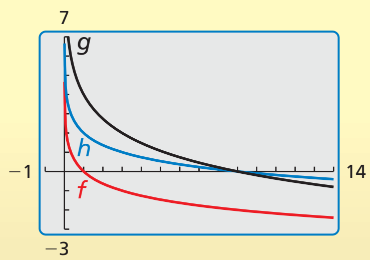
$g(x) = 2 \cdot h(x)$ Multiply the output by 2.

$= 2 \cdot (\log_{1/3} x + 2)$ Substitute $\log_{1/3} x + 2$ for $h(x)$.

$= 2 \log_{1/3} x + 4$ Distributive Property

▶ The transformed function is $g(x) = 2 \log_{1/3} x + 4$.

Check



Monitoring Progress

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7. Let the graph of g be a horizontal stretch by a factor of 3, followed by a translation 2 units up of the graph of $f(x) = e^{-x}$. Write a rule for g.

8. Let the graph of g be a reflection in the y-axis, followed by a translation 4 units to the left of the graph of $f(x) = \log x$. Write a rule for g.

6.4 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **WRITING** Given the function $f(x) = ab^{x-h} + k$, describe the effects of a, h, and k on the graph of the function.

2. **COMPLETE THE SENTENCE** The graph of $g(x) = \log_4(-x)$ is a reflection in the __________ of the graph of $f(x) = \log_4 x$.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, match the function with its graph. Explain your reasoning.

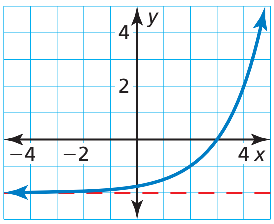
3. $f(x) = 2^{x+2} - 2$

4. $g(x) = 2^{x+2} + 2$

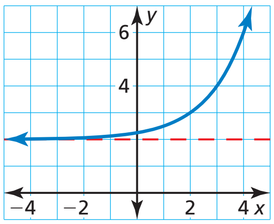
5. $h(x) = 2^{x-2} - 2$

6. $k(x) = 2^{x-2} + 2$

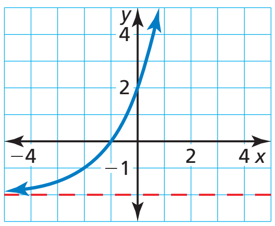
A.



B.



C.



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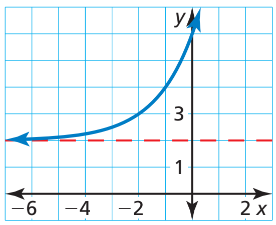


In Exercises 7–16, describe the transformation of *f* represented by *g*. Then graph each function. *(See Examples 1 and 2.)*

7. $f(x) = 3^x,\ g(x) = 3^x + 5$

8. $f(x) = 4^x,\ g(x) = 4^x - 8$

9. $f(x) = e^x,\ g(x) = e^x - 1$

10. $f(x) = e^x,\ g(x) = e^x + 4$

11. $f(x) = 2^x,\ g(x) = 2^{x-7}$

12. $f(x) = 5^x,\ g(x) = 5^{x+1}$

13. $f(x) = e^{-x},\ g(x) = e^{-x} + 6$

14. $f(x) = e^{-x},\ g(x) = e^{-x} - 9$

15. $f(x) = \left(\frac{1}{4}\right)^x,\ g(x) = \left(\frac{1}{4}\right)^{x-3} + 12$

16. $f(x) = \left(\frac{1}{3}\right)^x,\ g(x) = \left(\frac{1}{3}\right)^{x+2} - \frac{2}{3}$

In Exercises 17–24, describe the transformation of *f* represented by *g*. Then graph each function. *(See Example 3.)*

17. $f(x) = e^x,\ g(x) = e^{2x}$

18. $f(x) = e^x,\ g(x) = \frac{4}{3}e^x$

19. $f(x) = 2^x,\ g(x) = -2^{x-3}$

20. $f(x) = 4^x,\ g(x) = 4^{0.5x-5}$

21. $f(x) = e^{-x},\ g(x) = 3e^{-6x}$

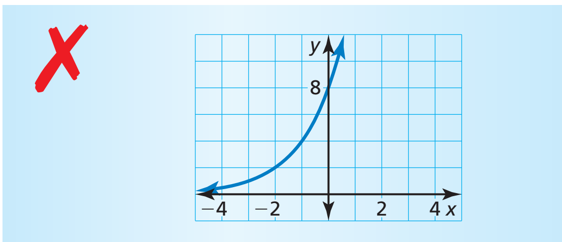
22. $f(x) = e^{-x},\ g(x) = e^{-5x} + 2$

23. $f(x) = \left(\frac{1}{2}\right)^x,\ g(x) = 6\left(\frac{1}{2}\right)^{x+5} - 2$

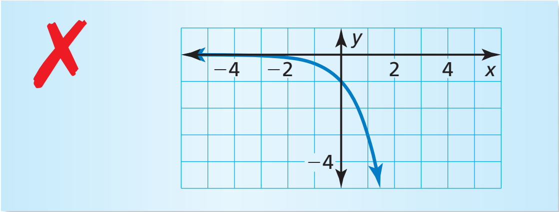
24. $f(x) = \left(\frac{3}{4}\right)^x,\ g(x) = -\left(\frac{3}{4}\right)^{x-7} + 1$

ERROR ANALYSIS **In Exercises 25 and 26, describe and correct the error in graphing the function.**

25. $f(x) = 2^x + 3$



26. $f(x) = 3^{-x}$



In Exercises 27–30, describe the transformation of *f* represented by *g*. Then graph each function. *(See Example 4.)*

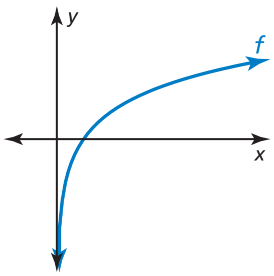
27. $f(x) = \log_4 x,\ g(x) = 3\log_4 x - 5$

28. $f(x) = \log_{1/3} x,\ g(x) = \log_{1/3}(-x) + 6$

29. $f(x) = \log_{1/5} x,\ g(x) = -\log_{1/5}(x - 7)$

30. $f(x) = \log_2 x,\ g(x) = \log_2(x + 2) - 3$

ANALYZING RELATIONSHIPS **In Exercises 31–34, match the function with the correct transformation of the graph of *f*. Explain your reasoning.**



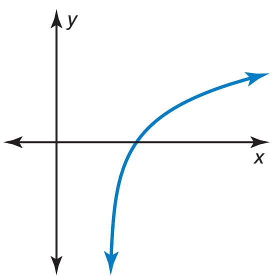
31. $y = f(x - 2)$

32. $y = f(x + 2)$

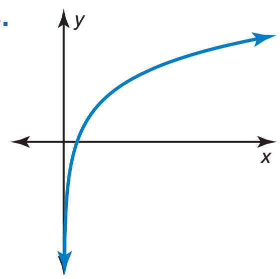
33. $y = 2f(x)$

34. $y = f(2x)$

A.



B.



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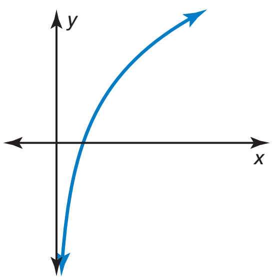

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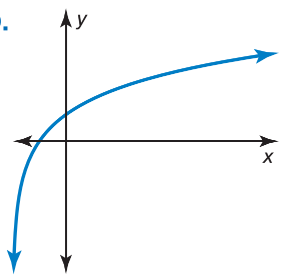


In Exercises 35–38, write a rule for *g* that represents the indicated transformations of the graph of *f*. *(See Example 5.)*

35. $f(x) = 5^x$; translation 2 units down, followed by a reflection in the y-axis

36. $f(x) = \left(\frac{2}{3}\right)^x$; reflection in the x-axis, followed by a vertical stretch by a factor of 6 and a translation 4 units left

37. $f(x) = e^x$; horizontal shrink by a factor of $\frac{1}{2}$, followed by a translation 5 units up

38. $f(x) = e^{-x}$; translation 4 units right and 1 unit down, followed by a vertical shrink by a factor of $\frac{1}{3}$

In Exercises 39–42, write a rule for *g* that represents the indicated transformation of the graph of *f*. *(See Example 6.)*

39. $f(x) = \log_6 x$; vertical stretch by a factor of 6, followed by a translation 5 units down

40. $f(x) = \log_5 x$; reflection in the x-axis, followed by a translation 9 units left

41. $f(x) = \log_{1/2} x$; translation 3 units left and 2 units up, followed by a reflection in the y-axis

42. $f(x) = \ln x$; translation 3 units right and 1 unit up, followed by a horizontal stretch by a factor of 8

JUSTIFYING STEPS **In Exercises 43 and 44, justify each step in writing a rule for *g* that represents the indicated transformations of the graph of *f*.**

43. $f(x) = \log_7 x$; reflection in the x-axis, followed by a translation 6 units down

$$h(x) = -f(x)$$
$$= -\log_7 x$$
$$g(x) = h(x) - 6$$
$$= -\log_7 x - 6$$

44. $f(x) = 8^x$; vertical stretch by a factor of 4, followed by a translation 1 unit up and 3 units left

$$h(x) = 4 \cdot f(x)$$
$$= 4 \cdot 8^x$$
$$g(x) = h(x + 3) + 1$$
$$= 4 \cdot 8^{x+3} + 1$$

USING STRUCTURE In Exercises 45–48, describe the transformation of the graph of f represented by the graph of g. Then give an equation of the asymptote.

45. $f(x) = e^x$, $g(x) = e^x + 4$

46. $f(x) = 3^x$, $g(x) = 3^{x-9}$

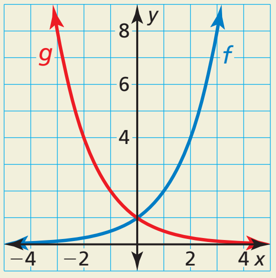
47. $f(x) = \ln x$, $g(x) = \ln(x + 6)$

48. $f(x) = \log_{1/5} x$, $g(x) = \log_{1/5} x + 13$

49. MODELING WITH MATHEMATICS The slope S of a beach is related to the average diameter d (in millimeters) of the sand particles on the beach by the equation $S = 0.159 + 0.118 \log d$. Describe the transformation of $f(d) = \log d$ represented by S. Then use the function to determine the slope of a beach for each sand type below.

Sand particle	Diameter (mm), d
fine sand	0.125
medium sand	0.25
coarse sand	0.5
very coarse sand	1

50. HOW DO YOU SEE IT? The graphs of $f(x) = b^x$ and $g(x) = \left(\frac{1}{b}\right)^x$ are shown for $b = 2$.



a. Use the graph to describe a transformation of the graph of f that results in the graph of g.

b. Does your answer in part (a) change when $0 < b < 1$? Explain.

51. MAKING AN ARGUMENT Your friend claims a single transformation of $f(x) = \log x$ can result in a function g whose graph never intersects the graph of f. Is your friend correct? Explain your reasoning.

52. THOUGHT PROVOKING Is it possible to transform the graph of $f(x) = e^x$ to obtain the graph of $g(x) = \ln x$? Explain your reasoning.

53. ABSTRACT REASONING Determine whether each statement is *always*, *sometimes*, or *never* true. Explain your reasoning.

a. A vertical translation of the graph of $f(x) = \log x$ changes the equation of the asymptote.

b. A vertical translation of the graph of $f(x) = e^x$ changes the equation of the asymptote.

c. A horizontal shrink of the graph of $f(x) = \log x$ does not change the domain.

d. The graph of $g(x) = ab^{x-h} + k$ does not intersect the x-axis.

54. PROBLEM SOLVING The amount P (in grams) of 100 grams of plutonium-239 that remains after t years can be modeled by $P = 100(0.99997)^t$.

a. Describe the domain and range of the function.

b. How much plutonium-239 is present after 12,000 years?

c. Describe the transformation of the function if the initial amount of plutonium were 550 grams.

d. Does the transformation in part (c) affect the domain and range of the function? Explain your reasoning.

55. CRITICAL THINKING Consider the graph of the function $h(x) = e^{-x-2}$. Describe the transformation of the graph of $f(x) = e^{-x}$ represented by the graph of h. Then describe the transformation of the graph of $g(x) = e^x$ represented by the graph of h. Justify your answers.

56. OPEN-ENDED Write a function of the form $y = ab^{x-h} + k$ whose graph has a y-intercept of 5 and an asymptote of $y = 2$.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Perform the indicated operation. *(Section 5.5)*

57. Let $f(x) = x^4$ and $g(x) = x^2$. Find $(fg)(x)$. Then evaluate the product when $x = 3$.

58. Let $f(x) = 4x^6$ and $g(x) = 2x^3$. Find $\left(\frac{f}{g}\right)(x)$. Then evaluate the quotient when $x = 5$.

59. Let $f(x) = 6x^3$ and $g(x) = 8x^3$. Find $(f + g)(x)$. Then evaluate the sum when $x = 2$.

60. Let $f(x) = 2x^2$ and $g(x) = 3x^2$. Find $(f - g)(x)$. Then evaluate the difference when $x = 6$.

6.1–6.4 What Did You Learn?

Core Vocabulary

exponential function, *p. 296*
exponential growth function, *p. 296*
growth factor, *p. 296*
asymptote, *p. 296*
exponential decay function, *p. 296*
decay factor, *p. 296*
natural base *e*, *p. 304*
logarithm of *y* with base *b*, *p. 310*
common logarithm, *p. 311*
natural logarithm, *p. 311*

Core Concepts

Section 6.1

Parent Function for Exponential Growth Functions, *p. 296*
Parent Function for Exponential Decay Functions, *p. 296*
Exponential Growth and Decay Models, *p. 297*
Compound Interest, *p. 299*

Section 6.2

The Natural Base *e*, *p. 304*
Natural Base Functions, *p. 305*
Continuously Compounded Interest, *p. 306*

Section 6.3

Definition of Logarithm with Base *b*, *p. 310*
Parent Graphs for Logarithmic Functions, *p. 313*

Section 6.4

Transforming Graphs of Exponential Functions, *p. 318*
Transforming Graphs of Logarithmic Functions, *p. 320*

Mathematical Practices

1. How did you check to make sure your answer was reasonable in Exercise 23 on page 300?
2. How can you justify your conclusions in Exercises 23–26 on page 307?
3. How did you monitor and evaluate your progress in Exercise 66 on page 315?

Study Skills

Forming a Weekly Study Group

- Select students who are just as dedicated to doing well in the math class as you are.
- Find a regular meeting place that has minimal distractions.
- Compare schedules and plan at least one time a week to meet, allowing at least 1.5 hours for study time.

6.1–6.4 Quiz

Tell whether the function represents *exponential growth* or *exponential decay*. Explain your reasoning. *(Sections 6.1 and 6.2)*

1. $f(x) = (4.25)^x$ **2.** $y = \left(\frac{3}{8}\right)^x$ **3.** $y = e^{0.6x}$ **4.** $f(x) = 5e^{-2x}$

Simplify the expression. *(Sections 6.2 and 6.3)*

5. $e^8 \bullet e^4$ **6.** $\frac{15e^3}{3e}$ **7.** $(5e^{4x})^3$

8. $e^{\ln 9}$ **9.** $\log_7 49^x$ **10.** $\log_3 81^{-2x}$

Rewrite the expression in exponential or logarithmic form. *(Section 6.3)*

11. $\log_4 1024 = 5$ **12.** $\log_{1/3} 27 = -3$ **13.** $7^4 = 2401$ **14.** $4^{-2} = 0.0625$

Evaluate the logarithm. If necessary, use a calculator and round your answer to three decimal places. *(Section 6.3)*

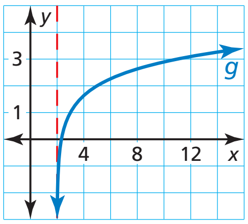
15. $\log 45$ **16.** $\ln 1.4$ **17.** $\log_2 32$

Graph the function and its inverse. *(Section 6.3)*

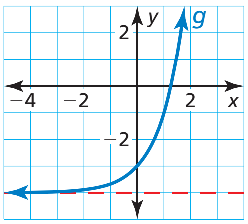
18. $f(x) = \left(\frac{1}{9}\right)^x$ **19.** $y = \ln(x - 7)$ **20.** $f(x) = \log_5(x + 1)$

The graph of *g* is a transformation of the graph of *f*. Write a rule for *g*. *(Section 6.4)*

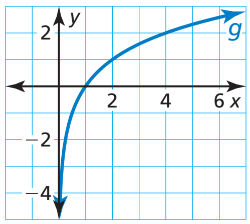
21. $f(x) = \log_3 x$



22. $f(x) = 3^x$



23. $f(x) = \log_{1/2} x$

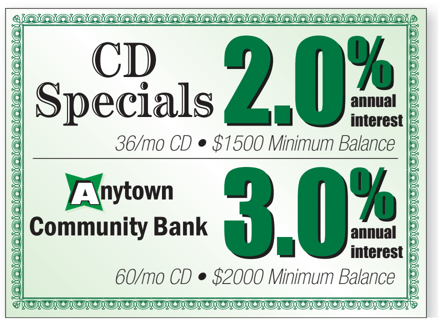


24. You purchase an antique lamp for \$150. The value of the lamp increases by 2.15% each year. Write an exponential model that gives the value *y* (in dollars) of the lamp *t* years after you purchased it. *(Section 6.1)*

25. A local bank advertises two certificate of deposit (CD) accounts that you can use to save money and earn interest. The interest is compounded monthly for both accounts. *(Section 6.1)*

a. You deposit the minimum required amounts in each CD account. How much money is in each account at the end of its term? How much interest does each account earn? Justify your answers.

b. Describe the benefits and drawbacks of each account.



26. The Richter scale is used for measuring the magnitude of an earthquake. The Richter magnitude *R* is given by $R = 0.67 \ln E + 1.17$, where *E* is the energy (in kilowatt-hours) released by the earthquake. Graph the model. What is the Richter magnitude for an earthquake that releases 23,000 kilowatt-hours of energy? *(Section 6.4)*

6.5 Properties of Logarithms

Essential Question

How can you use properties of exponents to derive properties of logarithms?

Let

$x = \log_b m$ and $y = \log_b n$.

The corresponding exponential forms of these two equations are

$b^x = m$ and $b^y = n$.

CONSTRUCTING VIABLE ARGUMENTS

To be proficient in math, you need to understand and use stated assumptions, definitions, and previously established results.

EXPLORATION 1 Product Property of Logarithms

Work with a partner. To derive the Product Property, multiply m and n to obtain

$mn = b^x b^y = b^{x+y}$.

The corresponding logarithmic form of $mn = b^{x+y}$ is $\log_b mn = x + y$. So,

$\log_b mn = \underline{\qquad\qquad}$. Product Property of Logarithms

EXPLORATION 2 Quotient Property of Logarithms

Work with a partner. To derive the Quotient Property, divide m by n to obtain

$\frac{m}{n} = \frac{b^x}{b^y} = b^{x-y}$.

The corresponding logarithmic form of $\frac{m}{n} = b^{x-y}$ is $\log_b \frac{m}{n} = x - y$. So,

$\log_b \frac{m}{n} = \underline{\qquad\qquad}$. Quotient Property of Logarithms

EXPLORATION 3 Power Property of Logarithms

Work with a partner. To derive the Power Property, substitute b^x for m in the expression $\log_b m^n$, as follows.

$\log_b m^n = \log_b (b^x)^n$ Substitute b^x for m.

$= \log_b b^{nx}$ Power of a Power Property of Exponents

$= nx$ Inverse Property of Logarithms

So, substituting $\log_b m$ for x, you have

$\log_b m^n = \underline{\qquad\qquad}$. Power Property of Logarithms

Communicate Your Answer

4. How can you use properties of exponents to derive properties of logarithms?

5. Use the properties of logarithms that you derived in Explorations 1–3 to evaluate each logarithmic expression.

a. $\log_4 16^3$ **b.** $\log_3 81^{-3}$

c. $\ln e^2 + \ln e^5$ **d.** $2 \ln e^6 - \ln e^5$

e. $\log_5 75 - \log_5 3$ **f.** $\log_4 2 + \log_4 32$

6.5 Lesson

What You Will Learn

- Use the properties of logarithms to evaluate logarithms.
- Use the properties of logarithms to expand or condense logarithmic expressions.
- Use the change-of-base formula to evaluate logarithms.

Core Vocabulary

Previous
base
properties of exponents

Properties of Logarithms

You know that the logarithmic function with base b is the inverse function of the exponential function with base b. Because of this relationship, it makes sense that logarithms have properties similar to properties of exponents.

Core Concept

Properties of Logarithms

Let b, m, and n be positive real numbers with $b \neq 1$.

Product Property $\log_b mn = \log_b m + \log_b n$

Quotient Property $\log_b \frac{m}{n} = \log_b m - \log_b n$

Power Property $\log_b m^n = n \log_b m$

STUDY TIP

These three properties of logarithms correspond to these three properties of exponents.

$a^m a^n = a^{m+n}$

$\frac{a^m}{a^n} = a^{m-n}$

$(a^m)^n = a^{mn}$

EXAMPLE 1 Using Properties of Logarithms

Use $\log_2 3 \approx 1.585$ and $\log_2 7 \approx 2.807$ to evaluate each logarithm.

a. $\log_2 \frac{3}{7}$ **b.** $\log_2 21$ **c.** $\log_2 49$

SOLUTION

a. $\log_2 \frac{3}{7} = \log_2 3 - \log_2 7$ Quotient Property

$\approx 1.585 - 2.807$ Use the given values of $\log_2 3$ and $\log_2 7$.

$= -1.222$ Subtract.

b. $\log_2 21 = \log_2(3 \cdot 7)$ Write 21 as $3 \cdot 7$.

$= \log_2 3 + \log_2 7$ Product Property

$\approx 1.585 + 2.807$ Use the given values of $\log_2 3$ and $\log_2 7$.

$= 4.392$ Add.

c. $\log_2 49 = \log_2 7^2$ Write 49 as 7^2.

$= 2 \log_2 7$ Power Property

$\approx 2(2.807)$ Use the given value $\log_2 7$.

$= 5.614$ Multiply.

COMMON ERROR

Note that in general $\log_b \frac{m}{n} \neq \frac{\log_b m}{\log_b n}$ and $\log_b mn \neq (\log_b m)(\log_b n)$.

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Use $\log_6 5 \approx 0.898$ and $\log_6 8 \approx 1.161$ to evaluate the logarithm.

1. $\log_6 \frac{5}{8}$ **2.** $\log_6 40$ **3.** $\log_6 64$ **4.** $\log_6 125$

Rewriting Logarithmic Expressions

You can use the properties of logarithms to expand and condense logarithmic expressions.

STUDY TIP

When you are expanding or condensing an expression involving logarithms, you can assume that any variables are positive.

EXAMPLE 2 Expanding a Logarithmic Expression

Expand $\ln \frac{5x^7}{y}$.

SOLUTION

$$\ln \frac{5x^7}{y} = \ln 5x^7 - \ln y \quad \text{Quotient Property}$$

$$= \ln 5 + \ln x^7 - \ln y \quad \text{Product Property}$$

$$= \ln 5 + 7 \ln x - \ln y \quad \text{Power Property}$$

EXAMPLE 3 Condensing a Logarithmic Expression

Condense $\log 9 + 3 \log 2 - \log 3$.

SOLUTION

$$\log 9 + 3 \log 2 - \log 3 = \log 9 + \log 2^3 - \log 3 \quad \text{Power Property}$$

$$= \log(9 \cdot 2^3) - \log 3 \quad \text{Product Property}$$

$$= \log \frac{9 \cdot 2^3}{3} \quad \text{Quotient Property}$$

$$= \log 24 \quad \text{Simplify.}$$

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Expand the logarithmic expression.

5. $\log_6 3x^4$

6. $\ln \frac{5}{12x}$

Condense the logarithmic expression.

7. $\log x - \log 9$

8. $\ln 4 + 3 \ln 3 - \ln 12$

Change-of-Base Formula

Logarithms with any base other than 10 or e can be written in terms of common or natural logarithms using the *change-of-base formula*. This allows you to evaluate any logarithm using a calculator.

Core Concept

Change-of-Base Formula

If a, b, and c are positive real numbers with $b \neq 1$ and $c \neq 1$, then

$$\log_c a = \frac{\log_b a}{\log_b c}.$$

In particular, $\log_c a = \frac{\log a}{\log c}$ and $\log_c a = \frac{\ln a}{\ln c}$.

EXAMPLE 4 Changing a Base Using Common Logarithms

ANOTHER WAY

In Example 4, $\log_3 8$ can be evaluated using natural logarithms.

$\log_3 8 = \frac{\ln 8}{\ln 3} \approx 1.893$

Notice that you get the same answer whether you use natural logarithms or common logarithms in the change-of-base formula.

Evaluate $\log_3 8$ using common logarithms.

SOLUTION

$$\log_3 8 = \frac{\log 8}{\log 3} \qquad \log_c a = \frac{\log a}{\log c}$$

$$\approx \frac{0.9031}{0.4771} \approx 1.893 \qquad \text{Use a calculator. Then divide.}$$

EXAMPLE 5 Changing a Base Using Natural Logarithms

Evaluate $\log_6 24$ using natural logarithms.

SOLUTION

$$\log_6 24 = \frac{\ln 24}{\ln 6} \qquad \log_c a = \frac{\ln a}{\ln c}$$

$$\approx \frac{3.1781}{1.7918} \approx 1.774 \qquad \text{Use a calculator. Then divide.}$$

EXAMPLE 6 Solving a Real-Life Problem

For a sound with intensity I (in watts per square meter), the loudness $L(I)$ of the sound (in decibels) is given by the function

$$L(I) = 10 \log \frac{I}{I_0}$$

where I_0 is the intensity of a barely audible sound (about 10^{-12} watts per square meter). An artist in a recording studio turns up the volume of a track so that the intensity of the sound doubles. By how many decibels does the loudness increase?

SOLUTION

Let I be the original intensity, so that $2I$ is the doubled intensity.

$$\text{increase in loudness} = L(2I) - L(I) \qquad \text{Write an expression.}$$

$$= 10 \log \frac{2I}{I_0} - 10 \log \frac{I}{I_0} \qquad \text{Substitute.}$$

$$= 10\left(\log \frac{2I}{I_0} - \log \frac{I}{I_0}\right) \qquad \text{Distributive Property}$$

$$= 10\left(\log 2 + \log \frac{I}{I_0} - \log \frac{I}{I_0}\right) \qquad \text{Product Property}$$

$$= 10 \log 2 \qquad \text{Simplify.}$$

▶ The loudness increases by 10 log 2 decibels, or about 3 decibels.

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Use the change-of-base formula to evaluate the logarithm.

9. $\log_5 8$ **10.** $\log_8 14$ **11.** $\log_{26} 9$ **12.** $\log_{12} 30$

13. WHAT IF? In Example 6, the artist turns up the volume so that the intensity of the sound triples. By how many decibels does the loudness increase?

6.5 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** To condense the expression $\log_3 2x + \log_3 y$, you need to use the ___________ Property of Logarithms.

2. **WRITING** Describe two ways to evaluate $\log_7 12$ using a calculator.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, use $\log_7 4 \approx 0.712$ and $\log_7 12 \approx 1.277$ to evaluate the logarithm. *(See Example 1.)*

3. $\log_7 3$
4. $\log_7 48$
5. $\log_7 16$
6. $\log_7 64$
7. $\log_7 \frac{1}{4}$
8. $\log_7 \frac{1}{3}$

In Exercises 9–12, match the expression with the logarithm that has the same value. Justify your answer.

9. $\log_3 6 - \log_3 2$
10. $2 \log_3 6$
11. $6 \log_3 2$
12. $\log_3 6 + \log_3 2$

A. $\log_3 64$

B. $\log_3 3$

C. $\log_3 12$

D. $\log_3 36$

In Exercises 13–20, expand the logarithmic expression. *(See Example 2.)*

13. $\log_3 4x$
14. $\log_8 3x$
15. $\log 10x^5$
16. $\ln 3x^4$
17. $\ln \frac{x}{3y}$
18. $\ln \frac{6x^2}{y^4}$
19. $\log_7 5\sqrt{x}$
20. $\log_5 \sqrt[3]{x^2 y}$

ERROR ANALYSIS In Exercises 21 and 22, describe and correct the error in expanding the logarithmic expression.

21. ✗ $\log_2 5x = (\log_2 5)(\log_2 x)$

22. ✗ $\ln 8x^3 = 3 \ln 8 + \ln x$

In Exercises 23–30, condense the logarithmic expression. *(See Example 3.)*

23. $\log_4 7 - \log_4 10$
24. $\ln 12 - \ln 4$
25. $6 \ln x + 4 \ln y$
26. $2 \log x + \log 11$
27. $\log_5 4 + \frac{1}{3} \log_5 x$
28. $6 \ln 2 - 4 \ln y$
29. $5 \ln 2 + 7 \ln x + 4 \ln y$
30. $\log_3 4 + 2 \log_3 \frac{1}{2} + \log_3 x$

31. **REASONING** Which of the following is *not* equivalent to $\log_5 \frac{y^4}{3x}$? Justify your answer.

Ⓐ $4 \log_5 y - \log_5 3x$

Ⓑ $4 \log_5 y - \log_5 3 + \log_5 x$

Ⓒ $4 \log_5 y - \log_5 3 - \log_5 x$

Ⓓ $\log_5 y^4 - \log_5 3 - \log_5 x$

32. **REASONING** Which of the following equations is correct? Justify your answer.

Ⓐ $\log_7 x + 2 \log_7 y = \log_7(x + y^2)$

Ⓑ $9 \log x - 2 \log y = \log \frac{x^9}{y^2}$

Ⓒ $5 \log_4 x + 7 \log_2 y = \log_6 x^5 y^7$

Ⓓ $\log_9 x - 5 \log_9 y = \log_9 \frac{x}{5y}$

In Exercises 33–40, use the change-of-base formula to evaluate the logarithm. *(See Examples 4 and 5.)*

33. $\log_4 7$

34. $\log_5 13$

35. $\log_9 15$

36. $\log_8 22$

37. $\log_6 17$

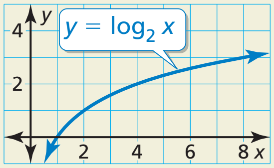
38. $\log_2 28$

39. $\log_7 \frac{3}{16}$

40. $\log_3 \frac{9}{40}$

41. MAKING AN ARGUMENT Your friend claims you can use the change-of-base formula to graph $y = \log_3 x$ using a graphing calculator. Is your friend correct? Explain your reasoning.

42. HOW DO YOU SEE IT? Use the graph to determine the value of $\frac{\log 8}{\log 2}$.

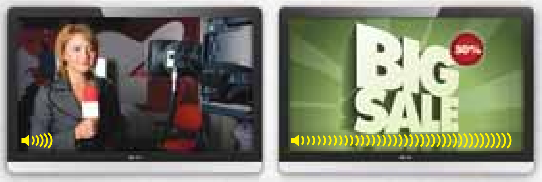


MODELING WITH MATHEMATICS **In Exercises 43 and 44, use the function $L(I)$ given in Example 6.**

43. The blue whale can produce sound with an intensity that is 1 million times greater than the intensity of the loudest sound a human can make. Find the difference in the decibel levels of the sounds made by a blue whale and a human. *(See Example 6.)*

44. The intensity of the sound of a certain television advertisement is 10 times greater than the intensity of the television program. By how many decibels does the loudness increase?

Intensity of Television Sound



During show: Intensity = I

During ad: Intensity = $10I$

45. REWRITING A FORMULA Under certain conditions, the wind speed s (in knots) at an altitude of h meters above a grassy plain can be modeled by the function

$$s(h) = 2 \ln 100h.$$

a. By what amount does the wind speed increase when the altitude doubles?

b. Show that the given function can be written in terms of common logarithms as

$$s(h) = \frac{2}{\log e}(\log h + 2).$$

46. THOUGHT PROVOKING Determine whether the formula

$$\log_b(M + N) = \log_b M + \log_b N$$

is true for all positive, real values of M, N, and b (with $b \neq 1$). Justify your answer.

47. USING STRUCTURE Use the properties of exponents to prove the change-of-base formula. (*Hint:* Let $x = \log_b a$, $y = \log_b c$, and $z = \log_c a$.)

48. CRITICAL THINKING Describe *three* ways to transform the graph of $f(x) = \log x$ to obtain the graph of $g(x) = \log 100x - 1$. Justify your answers.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Solve the inequality by graphing. *(Section 3.6)*

49. $x^2 - 4 > 0$

50. $2(x - 6)^2 - 5 \geq 37$

51. $x^2 + 13x + 42 < 0$

52. $-x^2 - 4x + 6 \leq -6$

Solve the equation by graphing the related system of equations. *(Section 3.5)*

53. $4x^2 - 3x - 6 = -x^2 + 5x + 3$

54. $-(x + 3)(x - 2) = x^2 - 6x$

55. $2x^2 - 4x - 5 = -(x + 3)^2 + 10$

56. $-(x + 7)^2 + 5 = (x + 10)^2 - 3$

6.6 Solving Exponential and Logarithmic Equations

Essential Question

How can you solve exponential and logarithmic equations?

EXPLORATION 1 Solving Exponential and Logarithmic Equations

Work with a partner. Match each equation with the graph of its related system of equations. Explain your reasoning. Then use the graph to solve the equation.

a. $e^x = 2$

b. $\ln x = -1$

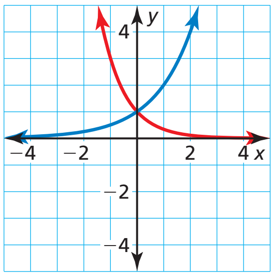
c. $2^x = 3^{-x}$

d. $\log_4 x = 1$

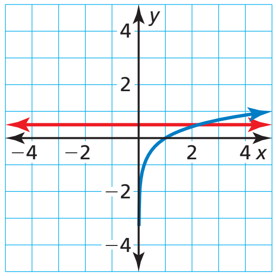
e. $\log_5 x = \frac{1}{2}$

f. $4^x = 2$

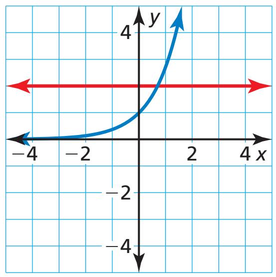
A.



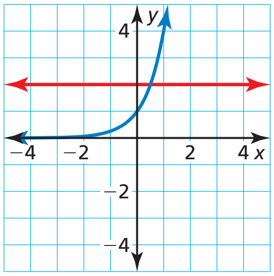
B.



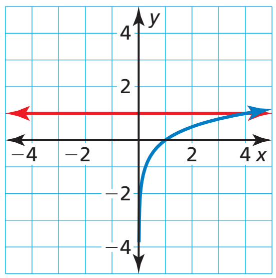
C.



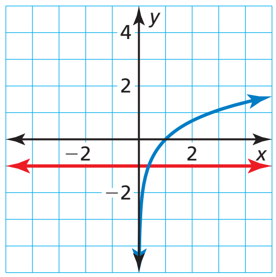
D.



E.



F.



MAKING SENSE OF PROBLEMS

To be proficient in math, you need to plan a solution pathway rather than simply jumping into a solution attempt.

EXPLORATION 2 Solving Exponential and Logarithmic Equations

Work with a partner. Look back at the equations in Explorations 1(a) and 1(b). Suppose you want a more accurate way to solve the equations than using a graphical approach.

a. Show how you could use a *numerical approach* by creating a table. For instance, you might use a spreadsheet to solve the equations.

b. Show how you could use an *analytical approach*. For instance, you might try solving the equations by using the inverse properties of exponents and logarithms.

Communicate Your Answer

3. How can you solve exponential and logarithmic equations?

4. Solve each equation using any method. Explain your choice of method.

a. $16^x = 2$

b. $2^x = 4^{2x+1}$

c. $2^x = 3^{x+1}$

d. $\log x = \frac{1}{2}$

e. $\ln x = 2$

f. $\log_3 x = \frac{3}{2}$

6.6 Lesson

Core Vocabulary

exponential equations, *p. 334*
logarithmic equations, *p. 335*

Previous
extraneous solution
inequality

What You Will Learn

- Solve exponential equations.
- Solve logarithmic equations.
- Solve exponential and logarithmic inequalities.

Solving Exponential Equations

Exponential equations are equations in which variable expressions occur as exponents. The result below is useful for solving certain exponential equations.

Core Concept

Property of Equality for Exponential Equations

Algebra If b is a positive real number other than 1, then $b^x = b^y$ if and only if $x = y$.

Example If $3^x = 3^5$, then $x = 5$. If $x = 5$, then $3^x = 3^5$.

The preceding property is useful for solving an exponential equation when each side of the equation uses the same base (or can be rewritten to use the same base). When it is not convenient to write each side of an exponential equation using the same base, you can try to solve the equation by taking a logarithm of each side.

EXAMPLE 1 Solving Exponential Equations

Solve each equation.

a. $100^x = \left(\frac{1}{10}\right)^{x-3}$ **b.** $2^x = 7$

SOLUTION

a.

$100^x = \left(\frac{1}{10}\right)^{x-3}$	Write original equation.
$(10^2)^x = (10^{-1})^{x-3}$	Rewrite 100 and $\frac{1}{10}$ as powers with base 10.
$10^{2x} = 10^{-x+3}$	Power of a Power Property
$2x = -x + 3$	Property of Equality for Exponential Equations
$x = 1$	Solve for x.

Check

$$100^1 \stackrel{?}{=} \left(\frac{1}{10}\right)^{1-3}$$

$$100 \stackrel{?}{=} \left(\frac{1}{10}\right)^{-2}$$

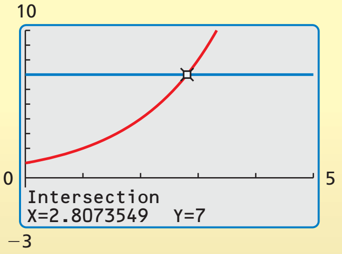
$$100 = 100 \checkmark$$

b.

$2^x = 7$	Write original equation.
$\log_2 2^x = \log_2 7$	Take $\log_2$ of each side.
$x = \log_2 7$	$\log_b b^x = x$
$x \approx 2.807$	Use a calculator.

Check

Enter $y = 2^x$ and $y = 7$ in a graphing calculator. Use the *intersect* feature to find the intersection point of the graphs. The graphs intersect at about (2.807, 7). So, the solution of $2^x = 7$ is about 2.807. ✓



An important application of exponential equations is *Newton's Law of Cooling*. This law states that for a cooling substance with initial temperature T_0, the temperature T after t minutes can be modeled by

$$T = (T_0 - T_R)e^{-rt} + T_R$$

where T_R is the surrounding temperature and r is the cooling rate of the substance.

LOOKING FOR STRUCTURE

Notice that Newton's Law of Cooling models the temperature of a cooling body by adding a constant function, T_R, to a decaying exponential function, $(T_0 - T_R)e^{-rt}$.

EXAMPLE 2 Solving a Real-Life Problem

You are cooking *aleecha*, an Ethiopian stew. When you take it off the stove, its temperature is 212°F. The room temperature is 70°F, and the cooling rate of the stew is $r = 0.046$. How long will it take to cool the stew to a serving temperature of 100°F?

SOLUTION

Use Newton's Law of Cooling with $T = 100$, $T_0 = 212$, $T_R = 70$, and $r = 0.046$.

$T = (T_0 - T_R)e^{-rt} + T_R$	Newton's Law of Cooling
$100 = (212 - 70)e^{-0.046t} + 70$	Substitute for T, T_0, T_R, and r.
$30 = 142e^{-0.046t}$	Subtract 70 from each side.
$0.211 \approx e^{-0.046t}$	Divide each side by 142.
$\ln 0.211 \approx \ln e^{-0.046t}$	Take natural log of each side.
$-1.556 \approx -0.046t$	$\ln e^x = \log_e e^x = x$
$33.8 \approx t$	Divide each side by -0.046.

▶ You should wait about 34 minutes before serving the stew.

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Solve the equation.

1. $2^x = 5$
2. $7^{9x} = 15$
3. $4e^{-0.3x} - 7 = 13$
4. **WHAT IF?** In Example 2, how long will it take to cool the stew to 100°F when the room temperature is 75°F?

Solving Logarithmic Equations

Logarithmic equations are equations that involve logarithms of variable expressions. You can use the next property to solve some types of logarithmic equations.

Core Concept

Property of Equality for Logarithmic Equations

Algebra If b, x, and y are positive real numbers with $b \neq 1$, then $\log_b x = \log_b y$ if and only if $x = y$.

Example If $\log_2 x = \log_2 7$, then $x = 7$. If $x = 7$, then $\log_2 x = \log_2 7$.

The preceding property implies that if you are given an equation $x = y$, then you can exponentiate each side to obtain an equation of the form $b^x = b^y$. This technique is useful for solving some logarithmic equations.

EXAMPLE 3 Solving Logarithmic Equations

Solve (a) $\ln(4x - 7) = \ln(x + 5)$ and (b) $\log_2(5x - 17) = 3$.

SOLUTION

a. $\ln(4x - 7) = \ln(x + 5)$ — Write original equation.

$4x - 7 = x + 5$ — Property of Equality for Logarithmic Equations

$3x - 7 = 5$ — Subtract x from each side.

$3x = 12$ — Add 7 to each side.

$x = 4$ — Divide each side by 3.

Check

$\ln(4 \cdot 4 - 7) \stackrel{?}{=} \ln(4 + 5)$

$\ln(16 - 7) \stackrel{?}{=} \ln 9$

$\ln 9 = \ln 9$ ✓

b. $\log_2(5x - 17) = 3$ — Write original equation.

$2^{\log_2(5x - 17)} = 2^3$ — Exponentiate each side using base 2.

$5x - 17 = 8$ — $b^{\log_b x} = x$

$5x = 25$ — Add 17 to each side.

$x = 5$ — Divide each side by 5.

Check

$\log_2(5 \cdot 5 - 17) \stackrel{?}{=} 3$

$\log_2(25 - 17) \stackrel{?}{=} 3$

$\log_2 8 \stackrel{?}{=} 3$

Because $2^3 = 8$, $\log_2 8 = 3$. ✓

Because the domain of a logarithmic function generally does not include all real numbers, be sure to check for extraneous solutions of logarithmic equations. You can do this algebraically or graphically.

EXAMPLE 4 Solving a Logarithmic Equation

Solve $\log 2x + \log(x - 5) = 2$.

SOLUTION

$\log 2x + \log(x - 5) = 2$ — Write original equation.

$\log[2x(x - 5)] = 2$ — Product Property of Logarithms

$10^{\log[2x(x - 5)]} = 10^2$ — Exponentiate each side using base 10.

$2x(x - 5) = 100$ — $b^{\log_b x} = x$

$2x^2 - 10x = 100$ — Distributive Property

$2x^2 - 10x - 100 = 0$ — Write in standard form.

$x^2 - 5x - 50 = 0$ — Divide each side by 2.

$(x - 10)(x + 5) = 0$ — Factor.

$x = 10$ or $x = -5$ — Zero-Product Property

▶ The apparent solution $x = -5$ is extraneous. So, the only solution is $x = 10$.

Check

$\log(2 \cdot 10) + \log(10 - 5) \stackrel{?}{=} 2$

$\log 20 + \log 5 \stackrel{?}{=} 2$

$\log 100 \stackrel{?}{=} 2$

$2 = 2$ ✓

$\log[2 \cdot (-5)] + \log(-5 - 5) \stackrel{?}{=} 2$

$\log(-10) + \log(-10) \stackrel{?}{=} 2$

Because $\log(-10)$ is not defined, -5 is not a solution. ✗

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Solve the equation. Check for extraneous solutions.

5. $\ln(7x - 4) = \ln(2x + 11)$

6. $\log_2(x - 6) = 5$

7. $\log 5x + \log(x - 1) = 2$

8. $\log_4(x + 12) + \log_4 x = 3$

Solving Exponential and Logarithmic Inequalities

Exponential inequalities are inequalities in which variable expressions occur as exponents, and *logarithmic inequalities* are inequalities that involve logarithms of variable expressions. To solve exponential and logarithmic inequalities algebraically, use these properties. Note that the properties are true for $\leq$ and $\geq$.

STUDY TIP

Be sure you understand that these properties of inequality are only true for values of $b > 1$.

Exponential Property of Inequality: If b is a positive real number greater than 1, then $b^x > b^y$ if and only if $x > y$, and $b^x < b^y$ if and only if $x < y$.

Logarithmic Property of Inequality: If b, x, and y are positive real numbers with $b > 1$, then $\log_b x > \log_b y$ if and only if $x > y$, and $\log_b x < \log_b y$ if and only if $x < y$.

You can also solve an inequality by taking a logarithm of each side or by exponentiating.

EXAMPLE 5 Solving an Exponential Inequality

Solve $3^x < 20$.

SOLUTION

$3^x < 20$ — Write original inequality.

$\log_3 3^x < \log_3 20$ — Take $\log_3$ of each side.

$x < \log_3 20$ — $\log_b b^x = x$

▶ The solution is $x < \log_3 20$. Because $\log_3 20 \approx 2.727$, the approximate solution is $x < 2.727$.

EXAMPLE 6 Solving a Logarithmic Inequality

Solve $\log x \leq 2$.

SOLUTION

Method 1 Use an algebraic approach.

$\log x \leq 2$ — Write original inequality.

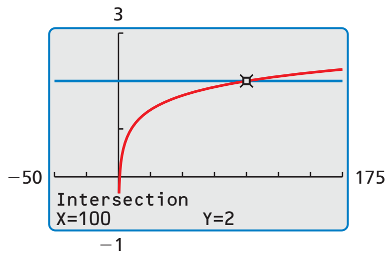
$10^{\log_{10} x} \leq 10^2$ — Exponentiate each side using base 10.

$x \leq 100$ — $b^{\log_b x} = x$

▶ Because $\log x$ is only defined when $x > 0$, the solution is $0 < x \leq 100$.

Method 2 Use a graphical approach.

Graph $y = \log x$ and $y = 2$ in the same viewing window. Use the *intersect* feature to determine that the graphs intersect when $x = 100$. The graph of $y = \log x$ is on or below the graph of $y = 2$ when $0 < x \leq 100$.



▶ The solution is $0 < x \leq 100$.

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Solve the inequality.

9. $e^x < 2$ **10.** $10^{2x-6} > 3$ **11.** $\log x + 9 < 45$ **12.** $2 \ln x - 1 > 4$

6.6 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** The equation $3^{x-1} = 34$ is an example of a(n) __________ equation.

2. **WRITING** Compare the methods for solving exponential and logarithmic equations.

3. **WRITING** When do logarithmic equations have extraneous solutions?

4. **COMPLETE THE SENTENCE** If b is a positive real number other than 1, then $b^x = b^y$ if and only if _________.

Monitoring Progress and Modeling with Mathematics

In Exercises 5–16, solve the equation. *(See Example 1.)*

5. $7^{3x+5} = 7^{1-x}$
6. $e^{2x} = e^{3x-1}$
7. $5^{x-3} = 25^{x-5}$
8. $6^{2x-6} = 36^{3x-5}$
9. $3^x = 7$
10. $5^x = 33$
11. $49^{5x+2} = \left(\frac{1}{7}\right)^{11-x}$
12. $512^{5x-1} = \left(\frac{1}{8}\right)^{-4-x}$
13. $7^{5x} = 12$
14. $11^{6x} = 38$
15. $3e^{4x} + 9 = 15$
16. $2e^{2x} - 7 = 5$

17. **MODELING WITH MATHEMATICS** The length ℓ (in centimeters) of a scalloped hammerhead shark can be modeled by the function

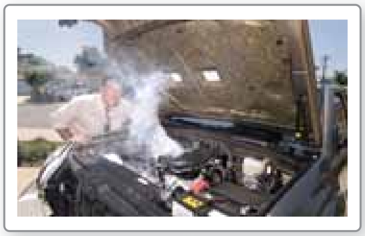
$$\ell = 266 - 219e^{-0.05t}$$

where t is the age (in years) of the shark. How old is a shark that is 175 centimeters long?

18. **MODELING WITH MATHEMATICS** One hundred grams of radium are stored in a container. The amount R (in grams) of radium present after t years can be modeled by $R = 100e^{-0.00043t}$. After how many years will only 5 grams of radium be present?

In Exercises 19 and 20, use Newton's Law of Cooling to solve the problem. *(See Example 2.)*

19. You are driving on a hot day when your car overheats and stops running. The car overheats at 280°F and can be driven again at 230°F. When it is 80°F outside, the cooling rate of the car is $r = 0.0058$. How long do you have to wait until you can continue driving?



20. You cook a turkey until the internal temperature reaches 180°F. The turkey is placed on the table until the internal temperature reaches 100°F and it can be carved. When the room temperature is 72°F, the cooling rate of the turkey is $r = 0.067$. How long do you have to wait until you can carve the turkey?

In Exercises 21–32, solve the equation. *(See Example 3.)*

21. $\ln(4x - 7) = \ln(x + 11)$
22. $\ln(2x - 4) = \ln(x + 6)$
23. $\log_2(3x - 4) = \log_2 5$
24. $\log(7x + 3) = \log 38$
25. $\log_2(4x + 8) = 5$
26. $\log_3(2x + 1) = 2$
27. $\log_7(4x + 9) = 2$
28. $\log_5(5x + 10) = 4$
29. $\log(12x - 9) = \log 3x$
30. $\log_6(5x + 9) = \log_6 6x$
31. $\log_2(x^2 - x - 6) = 2$
32. $\log_3(x^2 + 9x + 27) = 2$

In Exercises 33–40, solve the equation. Check for extraneous solutions. *(See Example 4.)*

33. $\log_2 x + \log_2(x - 2) = 3$

34. $\log_6 3x + \log_6(x - 1) = 3$

35. $\ln x + \ln(x + 3) = 4$

36. $\ln x + \ln(x - 2) = 5$

37. $\log_3 3x^2 + \log_3 3 = 2$

38. $\log_4(-x) + \log_4(x + 10) = 2$

39. $\log_3(x - 9) + \log_3(x - 3) = 2$

40. $\log_5(x + 4) + \log_5(x + 1) = 2$

ERROR ANALYSIS In Exercises 41 and 42, describe and correct the error in solving the equation.

41.

$$\log_3(5x - 1) = 4$$
$$3^{\log_3(5x-1)} = 4^3$$
$$5x - 1 = 64$$
$$5x = 65$$
$$x = 13$$

42.

$$\log_4(x + 12) + \log_4 x = 3$$
$$\log_4[(x + 12)(x)] = 3$$
$$4^{\log_4[(x+12)(x)]} = 4^3$$
$$(x + 12)(x) = 64$$
$$x^2 + 12x - 64 = 0$$
$$(x + 16)(x - 4) = 0$$
$$x = -16 \quad \text{or} \quad x = 4$$

43. PROBLEM SOLVING You deposit \$100 in an account that pays 6% annual interest. How long will it take for the balance to reach \$1000 for each frequency of compounding?

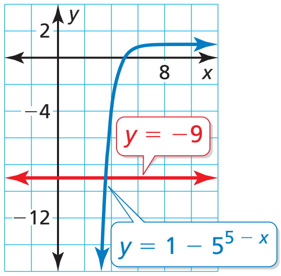
a. annual **b.** quarterly

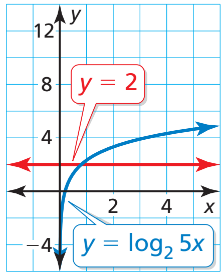
c. daily **d.** continuously

44. MODELING WITH MATHEMATICS The *apparent magnitude* of a star is a measure of the brightness of the star as it appears to observers on Earth. The apparent magnitude M of the dimmest star that can be seen with a telescope is $M = 5 \log D + 2$, where D is the diameter (in millimeters) of the telescope's objective lens. What is the diameter of the objective lens of a telescope that can reveal stars with a magnitude of 12?

45. ANALYZING RELATIONSHIPS Approximate the solution of each equation using the graph.

a. $1 - 5^{5-x} = -9$ **b.** $\log_2 5x = 2$





46. MAKING AN ARGUMENT Your friend states that a logarithmic equation cannot have a negative solution because logarithmic functions are not defined for negative numbers. Is your friend correct? Justify your answer.

In Exercises 47–54, solve the inequality. *(See Examples 5 and 6.)*

47. $9^x > 54$

48. $4^x \leq 36$

49. $\ln x \geq 3$

50. $\log_4 x < 4$

51. $3^{4x-5} < 8$

52. $e^{3x+4} > 11$

53. $-3 \log_5 x + 6 \leq 9$

54. $-4 \log_5 x - 5 \geq 3$

55. COMPARING METHODS Solve $\log_5 x < 2$ algebraically and graphically. Which method do you prefer? Explain your reasoning.

56. PROBLEM SOLVING You deposit \$1000 in an account that pays 3.5% annual interest compounded monthly. When is your balance at least \$1200? \$3500?

57. PROBLEM SOLVING An investment that earns a rate of return r doubles in value in t years, where $t = \dfrac{\ln 2}{\ln(1 + r)}$ and r is expressed as a decimal. What rates of return will double the value of an investment in less than 10 years?

58. PROBLEM SOLVING Your family purchases a new car for \$20,000. Its value decreases by 15% each year. During what interval does the car's value exceed \$10,000?

USING TOOLS In Exercises 59–62, use a graphing calculator to solve the equation.

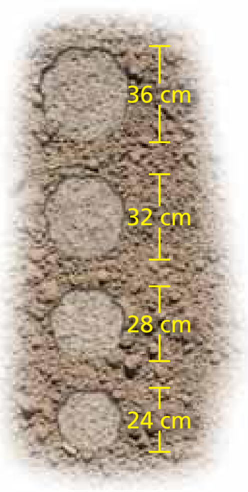
59. $\ln 2x = 3^{-x+2}$

60. $\log x = 7^{-x}$

61. $\log x = 3^{x-3}$

62. $\ln 2x = e^{x-3}$

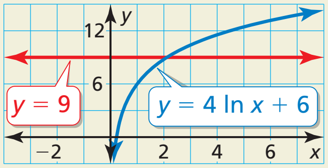
63. REWRITING A FORMULA A biologist can estimate the age of an African elephant by measuring the length of its footprint and using the equation $\ell = 45 - 25.7e^{-0.09a}$, where ℓ is the length (in centimeters) of the footprint and a is the age (in years).



a. Rewrite the equation, solving for a in terms of ℓ.

b. Use the equation in part (a) to find the ages of the elephants whose footprints are shown.

64. HOW DO YOU SEE IT? Use the graph to solve the inequality $4 \ln x + 6 > 9$. Explain your reasoning.



65. OPEN-ENDED Write an exponential equation that has a solution of $x = 4$. Then write a logarithmic equation that has a solution of $x = -3$.

66. THOUGHT PROVOKING Give examples of logarithmic or exponential equations that have one solution, two solutions, and no solutions.

CRITICAL THINKING In Exercises 67–72, solve the equation.

67. $2^{x+3} = 5^{3x-1}$

68. $10^{3x-8} = 2^{5-x}$

69. $\log_3(x - 6) = \log_9 2x$

70. $\log_4 x = \log_8 4x$

71. $2^{2x} - 12 \cdot 2^x + 32 = 0$

72. $5^{2x} + 20 \cdot 5^x - 125 = 0$

73. WRITING In Exercises 67–70, you solved exponential and logarithmic equations with different bases. Describe general methods for solving such equations.

74. PROBLEM SOLVING When X-rays of a fixed wavelength strike a material x centimeters thick, the intensity $I(x)$ of the X-rays transmitted through the material is given by $I(x) = I_0e^{-\mu x}$, where I_0 is the initial intensity and μ is a value that depends on the type of material and the wavelength of the X-rays. The table shows the values of μ for various materials and X-rays of medium wavelength.

Material	Aluminum	Copper	Lead
Value of μ	0.43	3.2	43

a. Find the thickness of aluminum shielding that reduces the intensity of X-rays to 30% of their initial intensity. (*Hint*: Find the value of x for which $I(x) = 0.3I_0$.)

b. Repeat part (a) for the copper shielding.

c. Repeat part (a) for the lead shielding.

d. Your dentist puts a lead apron on you before taking X-rays of your teeth to protect you from harmful radiation. Based on your results from parts (a)–(c), explain why lead is a better material to use than aluminum or copper.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Write an equation in point-slope form of the line that passes through the given point and has the given slope. *(Skills Review Handbook)*

75. $(1, -2)$; $m = 4$

76. $(3, 2)$; $m = -2$

77. $(3, -8)$; $m = -\frac{1}{3}$

78. $(2, 5)$; $m = 2$

Use finite differences to determine the degree of the polynomial function that fits the data. Then use technology to find the polynomial function. *(Section 4.9)*

79. $(-3, -50), (-2, -13), (-1, 0), (0, 1), (1, 2), (2, 15), (3, 52), (4, 125)$

80. $(-3, 139), (-2, 32), (-1, 1), (0, -2), (1, -1), (2, 4), (3, 37), (4, 146)$

81. $(-3, -327), (-2, -84), (-1, -17), (0, -6), (1, -3), (2, -32), (3, -189), (4, -642)$

6.7 Modeling with Exponential and Logarithmic Functions

Essential Question How can you recognize polynomial, exponential, and logarithmic models?

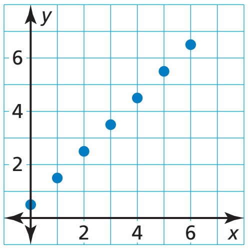
EXPLORATION 1 **Recognizing Different Types of Models**

Work with a partner. Match each type of model with the appropriate scatter plot. Use a regression program to find a model that fits the scatter plot.

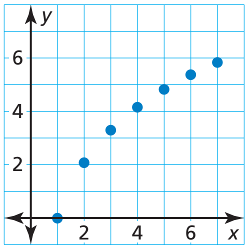
a. linear (positive slope) **b.** linear (negative slope) **c.** quadratic

d. cubic **e.** exponential **f.** logarithmic

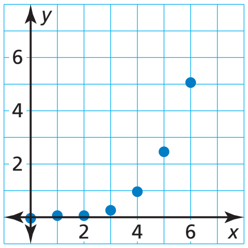
A.



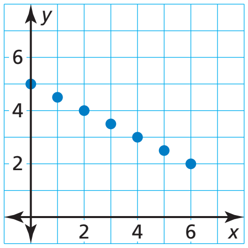
B.



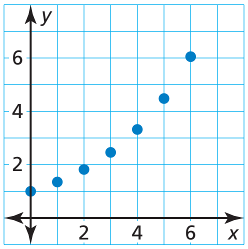
C.



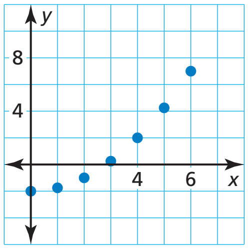
D.



E.



F.



USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technological tools to explore and deepen your understanding of concepts.

EXPLORATION 2 **Exploring Gaussian and Logistic Models**

Work with a partner. Two common types of functions that are related to exponential functions are given. Use a graphing calculator to graph each function. Then determine the domain, range, intercept, and asymptote(s) of the function.

a. Gaussian Function: $f(x) = e^{-x^2}$ **b.** Logistic Function: $f(x) = \dfrac{1}{1 + e^{-x}}$

Communicate Your Answer

3. How can you recognize polynomial, exponential, and logarithmic models?

4. Use the Internet or some other reference to find real-life data that can be modeled using one of the types given in Exploration 1. Create a table and a scatter plot of the data. Then use a regression program to find a model that fits the data.

6.7 Lesson

Core Vocabulary

Previous
finite differences
common ratio
point-slope form

What You Will Learn

- Classify data sets.
- Write exponential functions.
- Use technology to find exponential and logarithmic models.

Classifying Data

You have analyzed *finite differences* of data with equally-spaced inputs to determine what type of polynomial function can be used to model the data. For exponential data with equally-spaced inputs, the outputs are multiplied by a constant factor. So, consecutive outputs form a constant ratio.

EXAMPLE 1 Classifying Data Sets

Determine the type of function represented by each table.

a.

x	-2	-1	0	1	2	3	4
y	0.5	1	2	4	8	16	32

b.

x	-2	0	2	4	6	8	10
y	2	0	2	8	18	32	50

SOLUTION

a. The inputs are equally spaced. Look for a pattern in the outputs.

x	-2	-1	0	1	2	3	4
y	0.5	1	2	4	8	16	32

$\times 2$ $\quad\times 2$ $\quad\times 2$ $\quad\times 2$ $\quad\times 2$ $\quad\times 2$

- As x increases by 1, y is multiplied by 2. So, the common ratio is 2, and the data in the table represent an exponential function.

b. The inputs are equally spaced. The outputs do not have a common ratio. So, analyze the finite differences.

x	-2	0	2	4	6	8	10
y	2	0	2	8	18	32	50

$-2 \quad 2 \quad 6 \quad 10 \quad 14 \quad 18$ first differences

$4 \quad 4 \quad 4 \quad 4 \quad 4$ second differences

REMEMBER

First differences of linear functions are constant, second differences of quadratic functions are constant, and so on.

- The second differences are constant. So, the data in the table represent a quadratic function.

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Determine the type of function represented by the table. Explain your reasoning.

1.

x	0	10	20	30
y	15	12	9	6

2.

x	0	2	4	6
y	27	9	3	1

Writing Exponential Functions

You know that two points determine a line. Similarly, two points determine an exponential curve.

EXAMPLE 2 Writing an Exponential Function Using Two Points

Write an exponential function $y = ab^x$ whose graph passes through (1, 6) and (3, 54).

SOLUTION

Step 1 Substitute the coordinates of the two given points into $y = ab^x$.

$6 = ab^1$ Equation 1: Substitute 6 for *y* and 1 for *x*.

$54 = ab^3$ Equation 2: Substitute 54 for *y* and 3 for *x*.

Step 2 Solve for *a* in Equation 1 to obtain $a = \frac{6}{b}$ and substitute this expression for *a* in Equation 2.

$54 = \left(\frac{6}{b}\right)b^3$ Substitute $\frac{6}{b}$ for *a* in Equation 2.

$54 = 6b^2$ Simplify.

$9 = b^2$ Divide each side by 6.

$3 = b$ Take the positive square root because $b > 0$.

REMEMBER

You know that *b* must be positive by the definition of an exponential function.

Step 3 Determine that $a = \frac{6}{b} = \frac{6}{3} = 2$.

So, the exponential function is $y = 2(3^x)$.

Data do not always show an *exact* exponential relationship. When the data in a scatter plot show an *approximately* exponential relationship, you can model the data with an exponential function.

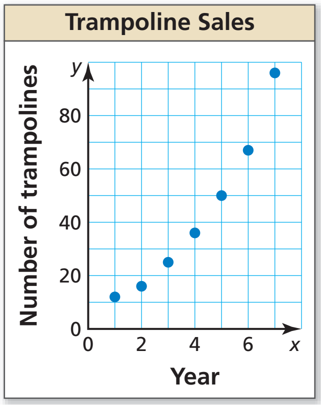
EXAMPLE 3 Finding an Exponential Model

A store sells trampolines. The table shows the numbers *y* of trampolines sold during the *x*th year that the store has been open. Write a function that models the data.

Year, *x*	Number of trampolines, *y*
1	12
2	16
3	25
4	36
5	50
6	67
7	96

SOLUTION

Step 1 Make a scatter plot of the data. The data appear exponential.



Step 2 Choose any two points to write a model, such as (1, 12) and (4, 36). Substitute the coordinates of these two points into $y = ab^x$.

$12 = ab^1$

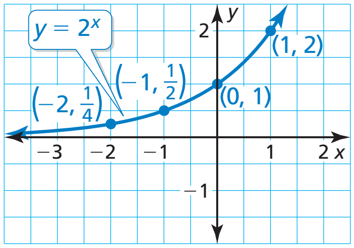
$36 = ab^4$

Solve for *a* in the first equation to obtain $a = \frac{12}{b}$. Substitute to obtain $b = \sqrt[3]{3} \approx 1.44$ and $a = \frac{12}{\sqrt[3]{3}} \approx 8.32$.

So, an exponential function that models the data is $y = 8.32(1.44)^x$.

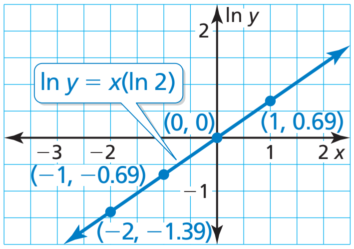
A set of more than two points (x, y) fits an exponential pattern if and only if the set of transformed points $(x, \ln y)$ fits a linear pattern.

Graph of points (x, y)



The graph is an exponential curve.

Graph of points $(x, \ln y)$



The graph is a line.

EXAMPLE 4 Writing a Model Using Transformed Points

Use the data from Example 3. Create a scatter plot of the data pairs $(x, \ln y)$ to show that an exponential model should be a good fit for the original data pairs (x, y). Then write an exponential model for the original data.

SOLUTION

Step 1 Create a table of data pairs $(x, \ln y)$.

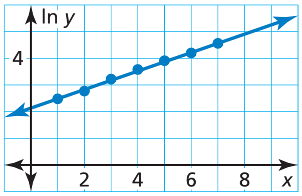
x	1	2	3	4	5	6	7
$\ln y$	2.48	2.77	3.22	3.58	3.91	4.20	4.56

LOOKING FOR STRUCTURE

Because the axes are x and $\ln y$, the point-slope form is rewritten as $\ln y - \ln y_1 = m(x - x_1)$. The slope of the line through (1, 2.48) and (7, 4.56) is

$$\frac{4.56 - 2.48}{7 - 1} \approx 0.35.$$

Step 2 Plot the transformed points as shown. The points lie close to a line, so an exponential model should be a good fit for the original data.



Step 3 Find an exponential model $y = ab^x$ by choosing any two points on the line, such as (1, 2.48) and (7, 4.56). Use these points to write an equation of the line. Then solve for y.

$$\ln y - 2.48 = 0.35(x - 1) \quad \text{Equation of line}$$

$$\ln y = 0.35x + 2.13 \quad \text{Simplify.}$$

$$y = e^{0.35x + 2.13} \quad \text{Exponentiate each side using base e.}$$

$$y = e^{0.35x}(e^{2.13}) \quad \text{Use properties of exponents.}$$

$$y = 8.41(1.42)^x \quad \text{Simplify.}$$

So, an exponential function that models the data is $y = 8.41(1.42)^x$.

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Write an exponential function $y = ab^x$ whose graph passes through the given points.

3. (2, 12), (3, 24) **4.** (1, 2), (3, 32) **5.** (2, 16), (5, 2)

6. WHAT IF? Repeat Examples 3 and 4 using the sales data from another store.

Year, x	1	2	3	4	5	6	7
Number of trampolines, y	15	23	40	52	80	105	140

Using Technology

You can use technology to find best-fit models for exponential and logarithmic data.

EXAMPLE 5 Finding an Exponential Model

Use a graphing calculator to find an exponential model for the data in Example 3. Then use this model and the models in Examples 3 and 4 to predict the number of trampolines sold in the eighth year. Compare the predictions.

SOLUTION

Enter the data into a graphing calculator and perform an exponential regression. The model is $y = 8.46(1.42)^x$.

```
ExpReg
 y=a*b^x
 a=8.457377971
 b=1.418848603
 r2=.9972445053
 r=.9986213023
```

Substitute $x = 8$ into each model to predict the number of trampolines sold in the eighth year.

Example 3: $y = 8.32(1.44)^8 \approx 154$

Example 4: $y = 8.41(1.42)^8 \approx 139$

Regression model: $y = 8.46(1.42)^8 \approx 140$

▶ The predictions are close for the regression model and the model in Example 4 that used transformed points. These predictions are less than the prediction for the model in Example 3.

EXAMPLE 6 Finding a Logarithmic Model

The atmospheric pressure decreases with increasing altitude. At sea level, the average air pressure is 1 atmosphere (1.033227 kilograms per square centimeter). The table shows the pressures p (in atmospheres) at selected altitudes h (in kilometers). Use a graphing calculator to find a logarithmic model of the form $h = a + b \ln p$ that represents the data. Estimate the altitude when the pressure is 0.75 atmosphere.

Weather balloons carry instruments that send back information such as wind speed, temperature, and air pressure.

Air pressure, p	1	0.55	0.25	0.12	0.06	0.02
Altitude, h	0	5	10	15	20	25

SOLUTION

Enter the data into a graphing calculator and perform a logarithmic regression. The model is $h = 0.86 - 6.45 \ln p$.

```
LnReg
 y=a+blnx
 a=.8626578705
 b=-6.447382985
 r2=.9925582287
 r=-.996272166
```

Substitute $p = 0.75$ into the model to obtain

$h = 0.86 - 6.45 \ln 0.75 \approx 2.7$.

▶ So, when the air pressure is 0.75 atmosphere, the altitude is about 2.7 kilometers.

Monitoring Progress

Help in English and Spanish at *BigIdeasMath.com*

7. Use a graphing calculator to find an exponential model for the data in Monitoring Progress Question 6.

8. Use a graphing calculator to find a logarithmic model of the form $p = a + b \ln h$ for the data in Example 6. Explain why the result is an error message.

6.7 Exercises

Dynamic Solutions available at *BigIdeasMath.com*

Vocabulary and Core Concept Check

1. **COMPLETE THE SENTENCE** Given a set of more than two data pairs (x, y), you can decide whether a(n) __________ function fits the data well by making a scatter plot of the points $(x, \ln y)$.

2. **WRITING** Given a table of values, explain how you can determine whether an exponential function is a good model for a set of data pairs (x, y).

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, determine the type of function represented by the table. Explain your reasoning. *(See Example 1.)*

3.

x	0	3	6	9	12	15
y	0.25	1	4	16	64	256

4.

x	−4	−3	−2	−1	0	1	2
y	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

5.

x	5	10	15	20	25	30
y	4	3	7	16	30	49

6.

x	−3	1	5	9	13
y	8	−3	−14	−25	−36

In Exercises 7–16, write an exponential function $y = ab^x$ whose graph passes through the given points. *(See Example 2.)*

7. (1, 3), (2, 12)

8. (2, 24), (3, 144)

9. (3, 1), (5, 4)

10. (3, 27), (5, 243)

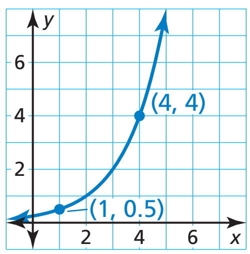
11. (1, 2), (3, 50)

12. (1, 40), (3, 640)

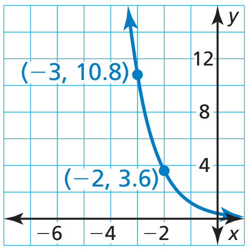
13. (−1, 10), (4, 0.31)

14. (2, 6.4), (5, 409.6)

15.



16.



ERROR ANALYSIS **In Exercises 17 and 18, describe and correct the error in determining the type of function represented by the data.**

17. ✗

x	0	1	2	3	4
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

×3 ×3 ×3 ×3

The outputs have a common ratio of 3, so the data represent a linear function.

18. ✗

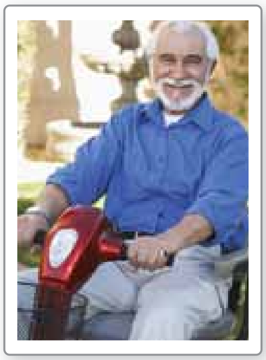
x	−2	−1	1	2	4
y	3	6	12	24	48

×2 ×2 ×2 ×2

The outputs have a common ratio of 2, so the data represent an exponential function.

19. **MODELING WITH MATHEMATICS** A store sells motorized scooters. The table shows the numbers y of scooters sold during the xth year that the store has been open. Write a function that models the data. *(See Example 3.)*

x	y
1	9
2	14
3	19
4	25
5	37
6	53
7	71



20. **MODELING WITH MATHEMATICS** The table shows the numbers y of visits to a website during the xth month. Write a function that models the data. Then use your model to predict the number of visits after 1 year.

x	1	2	3	4	5	6	7
y	22	39	70	126	227	408	735

In Exercises 21–24, determine whether the data show an exponential relationship. Then write a function that models the data.

21.

x	1	6	11	16	21
y	12	28	76	190	450

22.

x	−3	−1	1	3	5
y	2	7	24	68	194

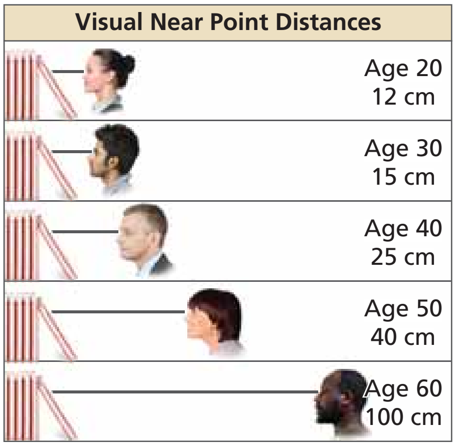
23.

x	0	10	20	30	40	50	60
y	66	58	48	42	31	26	21

24.

x	−20	−13	−6	1	8	15
y	25	19	14	11	8	6

25. **MODELING WITH MATHEMATICS** Your visual near point is the closest point at which your eyes can see an object distinctly. The diagram shows the near point y (in centimeters) at age x (in years). Create a scatter plot of the data pairs $(x, \ln y)$ to show that an exponential model should be a good fit for the original data pairs (x, y). Then write an exponential model for the original data. *(See Example 4.)*



26. **MODELING WITH MATHEMATICS** Use the data from Exercise 19. Create a scatter plot of the data pairs $(x, \ln y)$ to show that an exponential model should be a good fit for the original data pairs (x, y). Then write an exponential model for the original data.

In Exercises 27–30, create a scatter plot of the points $(x, \ln y)$ to determine whether an exponential model fits the data. If so, find an exponential model for the data.

27.

x	1	2	3	4	5
y	18	36	72	144	288

28.

x	1	4	7	10	13
y	3.3	10.1	30.6	92.7	280.9

29.

x	−13	−6	1	8	15
y	9.8	12.2	15.2	19	23.8

30.

x	−8	−5	−2	1	4
y	1.4	1.67	5.32	6.41	7.97

31. **USING TOOLS** Use a graphing calculator to find an exponential model for the data in Exercise 19. Then use the model to predict the number of motorized scooters sold in the tenth year. *(See Example 5.)*

32. **USING TOOLS** A doctor measures an astronaut's pulse rate y (in beats per minute) at various times x (in minutes) after the astronaut has finished exercising. The results are shown in the table. Use a graphing calculator to find an exponential model for the data. Then use the model to predict the astronaut's pulse rate after 16 minutes.

x	y
0	172
2	132
4	110
6	92
8	84
10	78
12	75

33. USING TOOLS An object at a temperature of 160°C is removed from a furnace and placed in a room at 20°C. The table shows the temperatures d (in degrees Celsius) at selected times t (in hours) after the object was removed from the furnace. Use a graphing calculator to find a logarithmic model of the form $t = a + b \ln d$ that represents the data. Estimate how long it takes for the object to cool to 50°C. *(See Example 6.)*

d	160	90	56	38	29	24
t	0	1	2	3	4	5

34. USING TOOLS The f-stops on a camera control the amount of light that enters the camera. Let s be a measure of the amount of light that strikes the film and let f be the f-stop. The table shows several f-stops on a 35-millimeter camera. Use a graphing calculator to find a logarithmic model of the form $s = a + b \ln f$ that represents the data. Estimate the amount of light that strikes the film when $f = 5.657$.

f	**s**
1.414	1
2.000	2
2.828	3
4.000	4
11.314	7

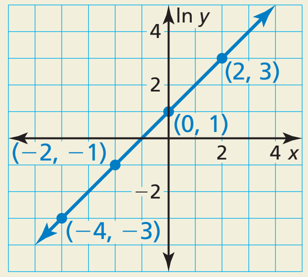
35. DRAWING CONCLUSIONS The table shows the average weight (in kilograms) of an Atlantic cod that is x years old from the Gulf of Maine.

Age, x	1	2	3	4	5
Weight, y	0.751	1.079	1.702	2.198	3.438

a. Show that an exponential model fits the data. Then find an exponential model for the data.

b. By what percent does the weight of an Atlantic cod increase each year in this period of time? Explain.

36. HOW DO YOU SEE IT? The graph shows a set of data points $(x, \ln y)$. Do the data pairs (x, y) fit an exponential pattern? Explain your reasoning.



37. MAKING AN ARGUMENT Your friend says it is possible to find a logarithmic model of the form $d = a + b \ln t$ for the data in Exercise 33. Is your friend correct? Explain.

38. THOUGHT PROVOKING Is it possible to write y as an exponential function of x? Explain your reasoning. (Assume p is positive.)

x	**y**
1	p
2	$2p$
3	$4p$
4	$8p$
5	$16p$

39. CRITICAL THINKING You plant a sunflower seedling in your garden. The height h (in centimeters) of the seedling after t weeks can be modeled by the *logistic function*

$$h(t) = \frac{256}{1 + 13e^{-0.65t}}.$$

a. Find the time it takes the sunflower seedling to reach a height of 200 centimeters.

b. Use a graphing calculator to graph the function. Interpret the meaning of the asymptote in the context of this situation.

Maintaining Mathematical Proficiency

Reviewing what you learned in previous grades and lessons

Tell whether x and y are in a proportional relationship. Explain your reasoning. *(Skills Review Handbook)*

40. $y = \frac{x}{2}$ **41.** $y = 3x - 12$ **42.** $y = \frac{5}{x}$ **43.** $y = -2x$

Identify the focus, directrix, and axis of symmetry of the parabola. Then graph the equation. *(Section 2.3)*

44. $x = \frac{1}{8}y^2$ **45.** $y = 4x^2$ **46.** $x^2 = 3y$ **47.** $y^2 = \frac{2}{5}x$

6.5–6.7 What Did You Learn?

Core Vocabulary

exponential equations, *p. 334*
logarithmic equations, *p. 335*

Core Concepts

Section 6.5

Properties of Logarithms, *p. 328*
Change-of-Base Formula, *p. 329*

Section 6.6

Property of Equality for Exponential Equations, *p. 334*
Property of Equality for Logarithmic Equations, *p. 335*

Section 6.7

Classifying Data, *p. 342*
Writing Exponential Functions, *p. 343*
Using Exponential and Logarithmic Regression, *p. 345*

Mathematical Practices

1. Explain how you used properties of logarithms to rewrite the function in part (b) of Exercise 45 on page 332.
2. How can you use cases to analyze the argument given in Exercise 46 on page 339?

Performance Task

Measuring Natural Disasters

In 2005, an earthquake measuring 4.1 on the Richter scale barely shook the city of Ocotillo, California, leaving virtually no damage. But in 1906, an earthquake with an estimated 8.2 on the same scale devastated the city of San Francisco. Does twice the measurement on the Richter scale mean twice the intensity of the earthquake?

To explore the answers to these questions and more, go to ***BigIdeasMath.com***.

6 Chapter Review

Dynamic Solutions available at *BigIdeasMath.com*

6.1 Exponential Growth and Decay Functions (pp. 295–302)

Tell whether the function $y = 3^x$ represents *exponential growth* or *exponential decay*. Then graph the function.

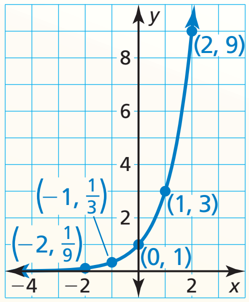
Step 1 Identify the value of the base. The base, 3, is greater than 1, so the function represents exponential growth.

Step 2 Make a table of values.

x	-2	-1	0	1	2
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

Step 3 Plot the points from the table.

Step 4 Draw, from *left to right*, a smooth curve that begins just above the x-axis, passes through the plotted points, and moves up to the right.



Tell whether the function represents exponential growth or exponential decay. Identify the percent increase or decrease. Then graph the function.

1. $f(x) = \left(\frac{1}{3}\right)^x$

2. $y = 5^x$

3. $f(x) = (0.2)^x$

4. You deposit \$1500 in an account that pays 7% annual interest. Find the balance after 2 years when the interest is compounded daily.

6.2 The Natural Base *e* (pp. 303–308)

Simplify each expression.

a. $\dfrac{18e^{13}}{2e^{7}} = 9e^{13-7} = 9e^{6}$

b. $(2e^{3x})^3 = 2^3(e^{3x})^3 = 8e^{9x}$

Simplify the expression.

5. $e^4 \cdot e^{11}$

6. $\dfrac{20e^3}{10e^6}$

7. $(-3e^{-5x})^2$

Tell whether the function represents *exponential growth* or *exponential decay*. Then graph the function.

8. $f(x) = \frac{1}{3}e^x$

9. $y = 6e^{-x}$

10. $y = 3e^{-0.75x}$

6.3 Logarithms and Logarithmic Functions (pp. 309–316)

Find the inverse of the function $y = \ln(x - 2)$.

$y = \ln(x - 2)$ Write original function.

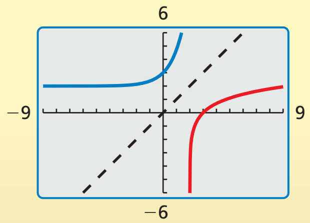
$x = \ln(y - 2)$ Switch x and y.

$e^x = y - 2$ Write in exponential form.

$e^x + 2 = y$ Add 2 to each side.

▶ The inverse of $y = \ln(x - 2)$ is $y = e^x + 2$.

Check



The graphs appear to be reflections of each other in the line $y = x$. ✓

Evaluate the logarithm.

11. $\log_2 8$ **12.** $\log_6 \frac{1}{36}$ **13.** $\log_5 1$

Find the inverse of the function.

14. $f(x) = 8^x$ **15.** $y = \ln(x - 4)$ **16.** $y = \log(x + 9)$

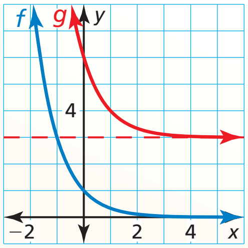
17. Graph $y = \log_{1/5} x$.

6.4 Transformations of Exponential and Logarithmic Functions (pp. 317–324)

Describe the transformation of $f(x) = \left(\frac{1}{3}\right)^x$ represented by $g(x) = \left(\frac{1}{3}\right)^{x-1} + 3$. Then graph each function.

Notice that the function is of the form $g(x) = \left(\frac{1}{3}\right)^{x-h} + k$, where $h = 1$ and $k = 3$.

▶ So, the graph of g is a translation 1 unit right and 3 units up of the graph of f.



Describe the transformation of f represented by g. Then graph each function.

18. $f(x) = e^{-x}$, $g(x) = e^{-5x} - 8$ **19.** $f(x) = \log_4 x$, $g(x) = \frac{1}{2}\log_4(x + 5)$

Write a rule for g.

20. Let the graph of g be a vertical stretch by a factor of 3, followed by a translation 6 units left and 3 units up of the graph of $f(x) = e^x$.

21. Let the graph of g be a translation 2 units down, followed by a reflection in the y-axis of the graph of $f(x) = \log x$.

6.5 Properties of Logarithms (pp. 327–332)

Expand $\ln \frac{12x^5}{y}$.

$$\ln \frac{12x^5}{y} = \ln 12x^5 - \ln y \quad \text{Quotient Property}$$

$$= \ln 12 + \ln x^5 - \ln y \quad \text{Product Property}$$

$$= \ln 12 + 5\ln x - \ln y \quad \text{Power Property}$$

Expand or condense the logarithmic expression.

22. $\log_8 3xy$ **23.** $\log 10x^3y$ **24.** $\ln \frac{3y}{x^5}$

25. $3\log_7 4 + \log_7 6$ **26.** $\log_2 12 - 2\log_2 x$ **27.** $2\ln x + 5\ln 2 - \ln 8$

Use the change-of-base formula to evaluate the logarithm.

28. $\log_2 10$ **29.** $\log_7 9$ **30.** $\log_{23} 42$

6.6 Solving Exponential and Logarithmic Equations (pp. 333–340)

Solve $\ln(3x - 9) = \ln(2x + 6)$.

$\ln(3x - 9) = \ln(2x + 6)$	Write original equation.
$3x - 9 = 2x + 6$	Property of Equality for Logarithmic Equations
$x - 9 = 6$	Subtract $2x$ from each side.
$x = 15$	Add 9 to each side.

Check

$\ln(3 \cdot 15 - 9) \stackrel{?}{=} \ln(2 \cdot 15 + 6)$

$\ln(45 - 9) \stackrel{?}{=} \ln(30 + 6)$

$\ln 36 = \ln 36$ ✓

Solve the equation. Check for extraneous solutions.

31. $5^x = 8$

32. $\log_3(2x - 5) = 2$

33. $\ln x + \ln(x + 2) = 3$

Solve the inequality.

34. $6^x > 12$

35. $\ln x \leq 9$

36. $e^{4x-2} \geq 16$

6.7 Modeling with Exponential and Logarithmic Functions (pp. 341–348)

Write an exponential function whose graph passes through (1, 3) and (4, 24).

Step 1 Substitute the coordinates of the two given points into $y = ab^x$.

$3 = ab^1$	Equation 1: Substitute 3 for y and 1 for x.
$24 = ab^4$	Equation 2: Substitute 24 for y and 4 for x.

Step 2 Solve for a in Equation 1 to obtain $a = \frac{3}{b}$ and substitute this expression for a in Equation 2.

$24 = \left(\frac{3}{b}\right)b^4$	Substitute $\frac{3}{b}$ for a in Equation 2.
$24 = 3b^3$	Simplify.
$8 = b^3$	Divide each side by 3.
$2 = b$	Take cube root of each side.

Step 3 Determine that $a = \frac{3}{b} = \frac{3}{2}$.

▶ So, the exponential function is $y = \frac{3}{2}(2^x)$.

Write an exponential model for the data pairs (x, y).

37. (3, 8), (5, 2)

38.

x	1	2	3	4
$\ln y$	1.64	2.00	2.36	2.72

39. A shoe store sells a new type of basketball shoe. The table shows the pairs sold s over time t (in weeks). Use a graphing calculator to find a logarithmic model of the form $s = a + b \ln t$ that represents the data. Estimate how many pairs of shoes are sold after 6 weeks.

Week, t	1	3	5	7	9
Pairs sold, s	5	32	48	58	65

6 Chapter Test

Graph the equation. State the domain, range, and asymptote.

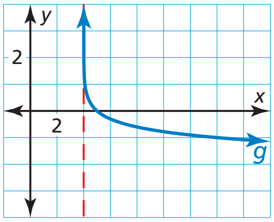
1. $y = \left(\frac{1}{2}\right)^x$

2. $y = \log_{1/5} x$

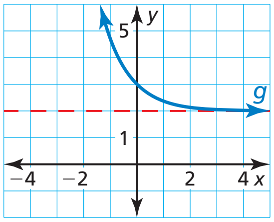
3. $y = 4e^{-2x}$

Describe the transformation of *f* represented by *g*. Then write a rule for *g*.

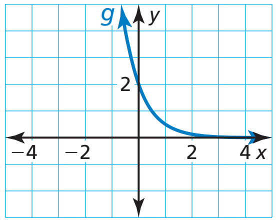
4. $f(x) = \log x$



5. $f(x) = e^x$



6. $f(x) = \left(\frac{1}{4}\right)^x$



Evaluate the logarithm. Use $\log_3 4 \approx 1.262$ and $\log_3 13 \approx 2.335$, if necessary.

7. $\log_3 52$

8. $\log_3 \frac{13}{9}$

9. $\log_3 16$

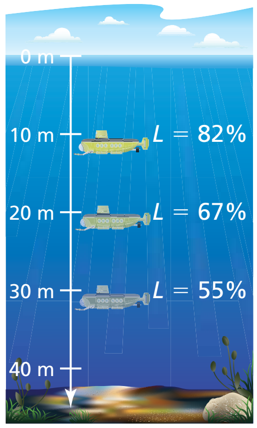
10. $\log_3 8 + \log_3 \frac{1}{2}$

11. Describe the similarities and differences in solving the equations $4^{5x-2} = 16$ and $\log_4(10x + 6) = 1$. Then solve each equation.

12. Without calculating, determine whether $\log_5 11$, $\frac{\log 11}{\log 5}$, and $\frac{\ln 11}{\ln 5}$ are equivalent expressions. Explain your reasoning.

13. The amount y of oil collected by a petroleum company drilling on the U.S. continental shelf can be modeled by $y = 12.263 \ln x - 45.381$, where y is measured in billions of barrels and x is the number of wells drilled. About how many barrels of oil would you expect to collect after drilling 1000 wells? Find the inverse function and describe the information you obtain from finding the inverse.

14. The percent L of surface light that filters down through bodies of water can be modeled by the exponential function $L(x) = 100e^{kx}$, where k is a measure of the murkiness of the water and x is the depth (in meters) below the surface.



a. A recreational submersible is traveling in clear water with a k-value of about -0.02. Write a function that gives the percent of surface light that filters down through clear water as a function of depth.

b. Tell whether your function in part (a) represents exponential growth or exponential decay. Explain your reasoning.

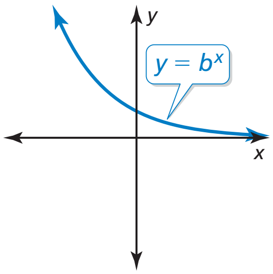
c. Estimate the percent of surface light available at a depth of 40 meters.

15. The table shows the values y (in dollars) of a new snowmobile after x years of ownership. Describe three different ways to find an exponential model that represents the data. Then write and use a model to find the year when the snowmobile is worth \$2500.

Year, *x*	0	1	2	3	4
Value, *y*	4200	3780	3402	3061.80	2755.60

6 Cumulative Assessment

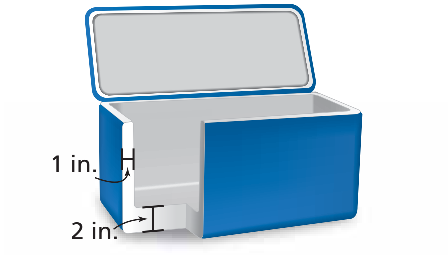
1. Select every value of b for the equation $y = b^x$ that could result in the graph shown.



1.08	0.94	e^2
2.04	$e^{-1/2}$	$\frac{5}{4}$

2. Your friend claims more interest is earned when an account pays interest compounded continuously than when it pays interest compounded daily. Do you agree with your friend? Justify your answer.

3. You are designing a rectangular picnic cooler with a length four times its width and height twice its width. The cooler has insulation that is 1 inch thick on each of the four sides and 2 inches thick on the top and bottom.



a. Let x represent the width of the cooler. Write a polynomial function T that gives the volume of the rectangular prism formed by the outer surfaces of the cooler.

b. Write a polynomial function C for the volume of the inside of the cooler.

c. Let I be a polynomial function that represents the volume of the insulation. How is I related to T and C?

d. Write I in standard form. What is the volume of the insulation when the width of the cooler is 8 inches?

4. What is the solution to the logarithmic inequality $-4 \log_2 x \geq -20$?

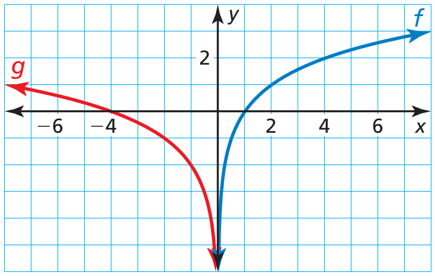
Ⓐ $x \leq 32$

Ⓑ $0 \leq x \leq 32$

Ⓒ $0 < x \leq 32$

Ⓓ $x \geq 32$

5. Describe the transformation of $f(x) = \log_2 x$ represented by the graph of g.



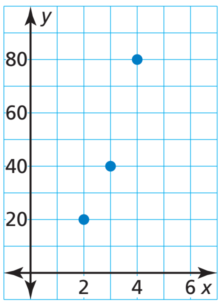
6. Let $f(x) = 2x^3 - 4x^2 + 8x - 1$, $g(x) = 2x - 3x^4 - 6x^3 + 5$, and $h(x) = -7 + x^2 + x$. Order the following functions from least degree to greatest degree.

A. $(f + g)(x)$ **B.** $(hg)(x)$

C. $(h - f)(x)$ **D.** $(fh)(x)$

7. Write an exponential model that represents each data set. Compare the two models.

a.



b.

x	2	3	4	5	6
y	4.5	13.5	40.5	121.5	364.5

8. Choose a method to solve each quadratic equation. Explain your choice of method.

a. $x^2 + 4x = 10$ **b.** $x^2 = -12$

c. $4(x - 1)^2 = 6x + 2$ **d.** $x^2 - 3x - 18 = 0$

9. At the annual pumpkin-tossing contest, contestants compete to see whose catapult will send pumpkins the longest distance. The table shows the horizontal distances y (in feet) a pumpkin travels when launched at different angles x (in degrees). Create a scatter plot of the data. Do the data show a linear, quadratic, or exponential relationship? Use technology to find a model for the data. Find the angle(s) at which a launched pumpkin travels 500 feet.

Angle (degrees), x	20	30	40	50	60	70
Distance (feet), y	372	462	509	501	437	323